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Recent Progress in the Development and Application of the Parabolic Equation

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PREFACE

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Four university professors, sponsored jointly by the Navy-ASEE Summer Faculty Research Program, the Naval Underwater Systems Center, and the Office of Naval Research, chose to visit NUSC during the summer of 1983 to participate in both the NUSC independent research project, "Finite Difference Solutions to Acoustic Wave Progation," and the ONR-sponsored research project, "Computational Acoustics." Research was performed jointly by the authors of this document. (cont'd)

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This technical document reports significant technical progress attained during the visit. Time did not permit the authors to carry out the necessary computations during the summer; however, as a continuing joint effort, computational results will be produced later to verify the validity of the theoretical developments. [illegible]

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1. INTRODUCTION

The Naval Underwater Systems Center (NUSC) sponsored Independent Research, "Finite-Difference Solutions to Acoustic Wave Propagations," has been successful. As a result to date, a useful product -- the implicit finite difference (IFD) computer software program for the solution of parabolic equations -- has been developed for research and application purposes. This software is now being used internationally in a number of research laboratories as well as universities. In relation to the development of the software package, the theoretical development attracted a number of internationally well-known scientists. In 1982, the Office of Naval Research (ONR) Mathematics Group, under the coordination of Dr. Richard L. Lau, awarded a research grant to NUSC to encourage technical collaboration with university scientists at the Yale University Center for Scientific Computation. These developments set the stage for four visiting scholars to spend the summer of 1983 performing research aimed at the solution of underwater acoustic wave propagation problems in all dimensions (mathematically, physically, and computationally).

This report is arranged in sections. Each section reports the technical accomplishments for a particular combination of authors. Some computations were performed by VAX 11/780 computers both at NUSC and at the Yale University Computer Science Department.

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2. A WIDE ANGLE THREE-DIMENSIONAL PARABOLIC WAVE EQUATION

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ABSTRACT: A simple extension of the standard two-dimensional parabolic wave equation to the three-dimensional case can be accomplished by retaining the angular derivative term. This extension is limited to dealing with small vertical angles of propagation. A new wide angle, three-dimensional partial differential equation is developed to predict the sound propagation in a three-dimensional ocean. This formulation is achieved by operator theory whose mathematical derivation is given in detail. The validity of the formulation is examined in full through discussion of approximation and multiple scale analysis.

INTRODUCTION

A three-dimensional parabolic equation (PE) was developed by Tappert[1] almost a decade ago. It was the same equation Tappert used to derive the two-dimensional parabolic wave equation, recognized as the standard small angle PE. Encountering the three-dimensional effect in the ocean, Baer[2] initiated the application of the primary three-dimensional PE to real problems. Recently Perkins and Baer implemented the Split-step algorithm[3] into a computer code to solve the three-dimensional PE. Application of this three-dimensional code demonstrated success in solving three-dimensional problems. Prior to Baer and Perkins' three-dimensional applications, Pierce[4] formulated a simplified three-dimensional parabolic wave equation expressing one spatial variable in terms of arc length. It is seen from the extension of the two-dimensional standard PE, the Tappert three-dimensional PE, implemented by Baer-Perkins for real applications (for simplicity we refer to the equation as the 3D PE), only handles the small angle propagation. So, Pierce has not pursued his development further. It is the purpose of this paper to report the development of the wide-angle three-dimensional PE, which accommodates the 3D PE. During the course of this wide angle development, a number of practical questions arose. We highlight the importance of these questions and try to answer these questions reasonably. The motivation of answering these questions led us to the formulation of the three-dimensional wide angle PE. These questions help to define the region of validity and suggest when and where the three-dimensional problem can be solved two-dimensionally. A formulation based on the operator theory is a starting

point; the complete detail is discussed. An analysis using the multiple scale technique is included to justify the operator formalism. A selected exact solution has been used by Schultz et al.[5] to discuss the validity of the formulation as well as the accuracy of the solution. In this paper, a simulated three-dimensional problem and an application are included to demonstrate the three-dimensional wide angle PE capability. All computations were performed on the VAX 11/780 computer using the Yale Sparse technique.[5]

OPERATOR FORMALISM

We begin from the three-dimensional Helmholtz equation for the spatially varying part of the acoustic pressure $p = p(r,\theta,z)$, written here in cylindrical coordinates (r,θ,z), i.e.,

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0 \quad . \tag{1}$$

The complex pressure is p times $e^{-i\omega t}$, where ω is acoustic frequency in rad/s. In Eq. (1), $k_0 = \omega/c_0$ and the index of refraction is $n = n(r,\theta,z) = c_0/c$, where $c = c(r,\theta,z)$, the oceanic sound speed, and c_0 is a reference sound speed. A thorough discussion of conditions and assumptions under which Eq. (1) applies to oceanic sound propagation has been given by Pierce.[6] Boundary conditions for Eq. (1) are to be specified at the ocean surface and bottom. The source term is omitted from the right side of Eq. (1) in anticipation of PE approximations that are valid away from the source, which is assumed near $r = 0$.

Following Tappert,[1] we let

$$p(r,\theta,z) = u(r,\theta,z)\ v(r) \quad , \tag{2}$$

in which the factor v(r) represents a rapidly-varying portion of the pressure and $u(r,\theta,z)$ is its modulation. Substituting Eq. (2) into Eq. (1) yields

$$\left[\frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{r} + \frac{2}{v}\frac{\partial v}{\partial r}\right)\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + k_0^2\, n^2\, u\right] v$$

$$+\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right] u = 0 \quad . \tag{3}$$

It follows from Eq. (3) that if an oscillatory function v is determined as a solution of

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + k_0^2\, v = 0 \quad , \tag{4}$$

then the u satisfies

$$\frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{r} + \frac{2}{v}\frac{\partial v}{\partial r}\right)\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + k_0^2(n^2 - 1)\, u = 0 \quad . \tag{5}$$

The outgoing-wave solution of Eq. (4) is

$$v(r) = H_0^{(1)}(k_0 r) \quad , \tag{6}$$

where $H_0^{(1)}$ is the Hankel function of zero-th order of the first kind. Since the parabolic approximation is desired for the solution at large distances from the source, it is appropriate to apply a farfield approximation, which is expressed by $k_0 r >> 1$. We defer until the next

section a discussion of the validity and the quantification of this assumption. For now we employ it to approximate Eq. (6) by an asymptotic expansion

$$v(r) \sim \left(\frac{2}{\pi k_0 r}\right)^{1/2} e^{i(k_0 r - \frac{\pi}{4})} , \quad k_0 r \to \infty \quad . \tag{7}$$

Using Eq. (7) in Eq. (5) gives

$$\left[\frac{\partial^2 u}{\partial r^2} + 2ik_0 \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + k_0^2(n^2 - 1)\right] u = 0 \quad . \tag{8}$$

If the first term in Eq. (8) is neglected, we obtain upon rearrangement a fundamental 3D PE, which is Eq. (?.7) of Ref. 1, i.e.,

$$\frac{\partial u}{\partial r} = \frac{ik_0}{2} [n^2(r,\theta,z) - 1]u + \frac{i}{2k_0}\frac{\partial^2 u}{\partial z^2} + \frac{i}{2k_0 r^2}\frac{\partial^2 u}{\partial \theta^2} \quad . \tag{9}$$

Neglecting the term $\frac{i}{2k_0 r^2}\frac{\partial^2 u}{\partial \theta^2}$ and regarding $n(r,\theta,z)$ as azimuthally independent the standard two-dimensional PE results, i.e.,

$$\frac{\partial u}{\partial r} = \frac{ik_0}{2} [n^2(r,z) - 1]u + \frac{i}{2k_0}\frac{\partial^2 u}{\partial z^2} \quad . \tag{10}$$

Equation (9) has been exploited in calculation of sound propagation through a mesoscale eddy.[2] If the last term in Eq. (9) is neglected but azimuthal dependence is retained in $n(r,\theta,z)$, then a simpler PE is obtained for which an efficient implementation has been demonstrated.[3] This equation is useful specifically in the absence of horizontal diffraction of acoustic energy, as

for example with weak azimuthal sound-speed variations and without azimuthal redirection of energy from boundary interactions. Finally, if all azimuthal dependence is neglected in Eq. (9), the usual two-dimensional PE remains. As is well known, Eq. (9) and its simplifications are valid for narrow vertical angles of propagation. In order to obtain a 3D PE appropriate for wider angles, we first employ an operator formalism.

We return to Eq. (8) and express it in operator form as

$$\left[\frac{\partial^2}{\partial r^2} + 2ik_0 \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + k_0^2(n^2 - 1)\right] u = 0 \quad . \qquad (11)$$

An approximation to Eq. (11) is made by factoring the operator as follows:

$$\left[\frac{\partial}{\partial r} + ik_0 - ik_0 Q\right] \left[\frac{\partial}{\partial r} + ik_0 + ik_0 Q\right] u = 0 \quad , \qquad (12)$$

where

$$Q = \left[1 + (n^2 - 1) + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + \frac{1}{k_0^2 r^2} \frac{\partial^2}{\partial \theta^2}\right]^{1/2} \quad . \qquad (13)$$

Equations (11) and (12) are not equivalent because the operators Q and $\partial/\partial r$ do not in general commute. However, provided these operators are in some sense nearly commutative, it is appropriate to regard Eq. (12) as a factorization approximation of Eq. (11). We make this approximation and will discuss later its validity. The solution of Eq. (12) consists of waves incoming and outgoing in the radial directions, and we neglect the incoming wave (the second factor in Eq. (12)), which is usual in the PE method. Therefore, the envelope $u(r,\theta,z)$ satisfies the formal equation

$$\frac{\partial u}{\partial r} + ik_0 u = ik_0 Qu \quad . \tag{14}$$

Determination of u requires some definition of the operator Q in Eq. (14).

We specify Q by first expressing it as

$$Q = [1 + X + Y]^{1/2} \quad , \tag{15}$$

and

$$X = (n^2 - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} \quad , \quad Y = \frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial \theta^2} \quad . \tag{16}$$

The fundamental 3D PE, Eq. (9), can be obtained by expanding the square root in Eq. (15) in a Taylor series and retaining only the linear terms in X and Y. Rather than a (linear) polynomial approximation for Q, we use a rational function approximation, i.e.,

$$Q \doteq \frac{1 + p_1 X + p_2 Y}{1 + q_1 X + q_2 Y} \quad , \tag{17}$$

where p_1, p_2, q_1, and q_2 are constants to be chosen. The interpretation of the fraction in Eq. (17) is premultiplication of the numerator by the inverse of the denominator. Thus when Eq. (17) is inserted in Eq. (14), the equation governing u becomes

$$\frac{\partial u}{\partial r} + ik_0 u = ik_0 [1 + q_1 X + q_2 Y]^{-1} [1 + p_1 X + p_2 Y]u \tag{18}$$

or, equivalently,

$$[1 + q_1 X + q_2 Y]\frac{\partial u}{\partial r} = ik_0 [(p_1 - q_1)X + (p_2 - q_2)Y]u \quad . \tag{19}$$

We note that when $q_1 = q_2 = 0$, Eq. (18) reduces to the narrow-angle 3D PE of Eq. (9) for the values $p_1 = p_2 = 1/2$, which are just those in the linear Taylor series for Q. For the two-dimensional problem $Y = 0$, rational-function approximations have been discussed.[7] In particular, the choices $p_1 = 3/4$, $q_1 = 1/4$ for the two-dimensional case $p_2 = q_2 = 0$ have been suggested by Claerbout[8] for wider-angle propagation. These values are precisely those necessary for an approximation to Q in Eq. (17) correct to quadratic terms in X. The analogous result for the three-dimensional case is found by squaring Eq. (17) and matching coefficients of X, Y, X^2, Y^2, and XY. It can be shown that the resulting five equations are satisfied by the four choices $p_1 = p2 = 3/4$ and $q_1 = q_2 = 1/4$. Thus, these values give a rational-function approximation to Q correct to second order in the operators X and Y. We use them in this paper to specify a wider-angle 3D PE from Eq. (19), i.e.,

$$\left[1 + \frac{1}{4}(n^2 - 1) + \frac{1}{4k_o^2}\frac{\partial^2}{\partial z^2} + \frac{1}{4(k_or)^2}\frac{1}{\partial\theta^2}\right]\frac{\partial u}{\partial r}$$

$$= \frac{ik_o}{2}\left[(n^2 - 1) + \frac{1}{k_o^2}\frac{\partial^2}{\partial z^2} + \frac{1}{(k_or)^2}\frac{\partial^2}{\partial\theta^2}\right]u \quad . \tag{20}$$

Neglecting the terms involving $\frac{1}{(k_or)^2}\frac{\partial^2}{\partial\theta^2}$ and regarding $n(r,\theta,z)$ as azimuthally independent, the two-dimensional wide angle PE results in the sense of Claerbout

$$\left[1 + \frac{1}{4}(n^2(r,z) - 1) + \frac{1}{4k_o^2}\frac{\partial^2}{\partial z^2}\right]\frac{\partial u}{\partial r}$$

$$= \frac{ik_0}{2}\left[(n^2(r,z) - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right] u \quad . \tag{21}$$

Note that Eq. (20) is a third-order partial differential equation and a discretized version has been analyzed for numerical stability.[9] Other choices for the parameters p_1, p_2, q_1, and q_2 have also been investigated.[10]

DISCUSSION OF APPROXIMATIONS

The wide-angle 3D PE of Eq. (20) was derived subject to a number of assumptions and approximations. The principal advantages and limitations common to all PE approximations are discussed in Ref. 1 (see also Ref 11). For applications, we are particularly interested in determination of limitations on oceanic ranges where Eq. (20) is appropriate and where the azimuthal-derivative terms in Eq. (20) are significant. We focus here on three of the assumptions used in the preceding section: farfield, factorization, and rational-function approximations. Our discussion leads to suggestive, rather than rigorous, conditions specifying range intervals where Eq. (20) or simplifications of it should be employed. These conditions are supported by arguments in this section; asymptotic derivations and numerical results will follow.

We turn first to the farfield approximation. Some indication of the range beyond which this approximation applies is very desirable in order to

provide an estimate for the minimum range of applicability for the 3D PE. Denote by $\bar{v}(r)$ the right side of Eq. (7), i.e., the leading term in the asymptotic expansion for $v(r)$ in Eq. (6) for large k_0r. Suppose we choose to regard $\bar{v}$ as an acceptable approximation to v if the relative difference in their moduli is less than some tolerance δ, i.e., if

$$\Big| |v(r)| - |\bar{v}(r)| \Big| \cdot |\bar{v}(r)|^{-1} \leq \delta \quad . \tag{22}$$

This condition focuses only on differences in modulus, rather than including differences in phase, which are of less interest here. Now it is known[12] that

$$|v(r)| \sim |\bar{v}(r)| \left\{1 - \frac{1}{16(k_0r)^2} + O\left[\frac{1}{(k_0r)^4}\right]\right\}, \; k_0r \to \infty \; . \tag{23}$$

In Eq. (23), the terms in braces ({ }) alternate in sign and have the property that the remainder, after retaining any number of terms, is no bigger than the first term neglected. From Eq. (23) it follows that the $\bar{v}$ is regarded as acceptably approximating v if

$$k_0r \geq \frac{1}{4\sqrt{\delta}} \quad . \tag{24}$$

In terms of acoustic frequency f, Eq. (24) requires range r to satisfy

$$r \geq r_f = \frac{c_0}{8\pi f\sqrt{\delta}} \tag{25}$$

in which r_f is the minimum range for the farfield approximation to apply. For example, suppose $\delta = 0.01$, corresponding to differences between v and $\bar{v}$

being bounded by 1%. Then as f increases from 10 to 200 Hz, r_f decreases from about 60 m to about 3 m. An alternative expression for this case is $k_o r \geq 2.5$.

Adoption of a criterion such as the above suggests neglect of terms of appropriately small estimated size in the governing equations. For instance, the approximation $\bar{v}(r)$ satisfies

$$\frac{d^2\bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} + k_o^2\,\bar{v}\left[1 - \frac{1}{4(k_o r)^2}\right] = 0 \quad , \tag{26}$$

rather than Eq. (4) satisfied by v(r). If Eqs. (24) and (25) hold, then the last term in Eq. (26) is no bigger than 4δ. Thus, the approximation of v by $\bar{v}$ is tantamount to neglecting this term, which for $\delta = 0.01$ is of relative magnitude no bigger than 4δ. This behavior is, of course, typical of a regular perturbation for which neglect of a term of some small size produces an error of comparable size in the solution. It follows that unless any term in a governing equation is capable of producing a singular-perturbation effect, it is apparently consistent to ignore the term if its relative size is no bigger than about 4%. An immediate application of this criterion is in the simplification of the coefficient of $\partial u/\partial r$ in Eq. (5). Using Eq. (7) and the result that the asymptotic expansion of dv/dr is the derivative of the expansion for v, it is easily shown that the coefficient of $\partial u/\partial r$ in Eq. (8) is multiplied by $(1 + 1/8(k_o r)^2)$. However, inequality (24) means that this factor in the farfield approximation is no bigger than $(1 + 2\delta.)$ Thus, it is apparently consistent to ignore this factor in the farfield

approximation. The inequality (24) with $\delta = 0.01$ (for instance) ensures that no more than 4% error is committed in terms of both the u and v equations in the farfield approximation and that the error in the modulus of $\bar{v}$ is even smaller. It can be shown by using another asymptotic expansion that the error in the phase of $\bar{v}$ is actually of comparable magnitude.

As employed in this paper, the rational-function approximation to the square-root operator Q is given by Eq. (17) with $p_1 = p_2 = 3/4$ and $q_1 = q_2 = 1/4$. For convenience this is rewritten as

$$Q = \frac{1 + (3/4)Z}{1 + (1/4)Z} , \tag{27}$$

where

$$Z = X + Y = n^2 - 1 + \frac{1}{k_o^2}\frac{\partial^2}{\partial z^2} + \frac{1}{(k_o r)^2}\frac{\partial^2}{\partial \theta^2} . \tag{28}$$

As mentioned previously, a primary advantage of this approximation is that it is correct to second order in Z, i.e.,

$$\begin{aligned} Q &= (1 + \tfrac{1}{4}Z)^{-1} (1 + \tfrac{3}{4}Z) \\ &= 1 + \tfrac{1}{2}Z - \tfrac{1}{8}Z^2 + O(Z^3) . \end{aligned} \tag{29}$$

An alternative expression of this fact is that the only other condition needed for Eqs. (11) and (12) to be identical, in addition to commutativity of Q and $\partial/\partial r$, is

$$Q^2 - 1 = Z . \tag{30}$$

From Eq. (29) it can be shown that Eq. (30a) holds to terms of $O(Z^3)$. Next, we note that

$$Z^2 = (n^2 - 1)^2 + \frac{1}{k_0^4}\frac{\partial^4}{\partial z^4} + \frac{1}{(k_0 r)^4}\frac{\partial^4}{\partial \theta^4} + \frac{2}{k_0^4 r^2}\frac{\partial^4}{\partial \theta^2 \partial z^2}$$

$$+ \frac{2(n^2 - 1)}{k_0^2}\left[\frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] + \frac{2}{k_0^2}\left[\frac{\partial^2}{\partial z^2}(n^2 - 1) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(n^2 - 1)\right] , \qquad (31)$$

and it follows that the effects of the terms in Eq. (31) are included by Eq. (27). Thus, even though Eq. (20) explicitly contains no fourth-order z and θ derivatives the effects of fourth-order derivatives in Eq. (31) are in some sense incorporated properly into Eq. (20). On the other hand, Eq. (20) does not contain the effects of any sixth-order derivatives, such as those appearing in Z^3, or similar terms like $(n^2 - 1)^3$ or $(n^2 - 1)^2 (k_0)^{-2} (\partial^2/\partial^2 z)$. The comparison of terms neglected with those retained is most easily accomplished by scaling and asymptotic expansions such as those in the next section. The purposes of the limited discussion here are to indicate which types of terms are modeled correctly by the approximation Eq. (27) and to provide a basis for examining the factorization approximation.

The factorization approximation is exact when Q from Eq. (13) and $\partial/\partial r$ commute. Since this is not true in general, Eq. (12) can be expanded to yield Eq. (11) with the additional term

$$ik_0\left[\frac{\partial}{\partial r}Q - Q\frac{\partial}{\partial r}\right] u = \quad u \qquad (32)$$

on the left side. To appraise the neglect of this unbounded operator, we compare its terms with those retained in the equation for u. We use the expression Eq. (27) for Q, but other definitions from Eq. (29) or other parameter choices in Eq. (17) could be treated similarly. In view of Eq. (29), it follows from Eq. (32) that

$$\xi u = \xi^{(1)}u + \xi^{(2)}u \quad , \tag{33}$$

where

$$\xi^{(1)} = \frac{ik_0}{2}\left[\frac{\partial}{\partial r}Z - Z\frac{\partial}{\partial r}\right] \quad , \tag{34}$$

and

$$\xi^{(2)} = -\frac{ik_0}{8}\left[\frac{\partial}{\partial r}Z^2 - Z^2\frac{\partial}{\partial r}\right] \quad . \tag{35}$$

Using Eq. (28) we find that the leading term $^{(1)}$ has the form

$$\xi^{(1)} = ik_0 n\frac{\partial n}{\partial r} - \frac{i}{k_0 r^3}\frac{\partial^2}{\partial\theta^2} \quad . \tag{36}$$

As with the farfield approximation, a comparison should be made here of terms neglected (the largest of which are $^{(1)}u$) with those retained [in Eq. (11)]. It follows that the factorization relies on $k_0 n\partial n/\partial r$ being small compared to $k_0^2(n^2-1)$, and $(k_0r^3)^{-1}(\partial^2u/\partial\theta^2)$ being small compared to $r^{-2}(\partial^2u/\partial\theta^2)$. Since n is close to one, these conditions are

$$k_0^{-1}(n^2-1)^{-1}(\partial n/\partial r) \text{ small} \tag{37}$$

and

$$(k_0r)^{-1} \text{ small} \quad . \tag{38}$$

The condition [Eq. (37)] of sufficiently small range variation of the index of refraction is anticipated from analysis of the validity of the two-dimensional PE[1]. Equation (38) is related to the justification of the farfield approximation since inequality (24) can be written as $(k_0 r)^{-1} \leq 4\delta$. Thus, for δ sufficiently small and Eq. (37) valid, both farfield and factorization approximations are satisifed. Furthermore, Eq. (38) and Eq. (37) can be quantified by recalling the argument following Eq. (26). It is apparently consistent to ignore the effect of the second term on the right of Eq. (34) if

$$(k_0 r)^{-1} \leq 4\delta \quad . \tag{39}$$

Here we have already neglected terms in governing equations of this relative magnitude. Equivalently, for $k_0 r \geq (4\ \delta)^{-1} = 25$ (when $\delta = 0.01$), the second term in Eq. (36) must be ignored; for f = 10 Hz (or 200 Hz), this corresponds to ranges bigger than about 600 m (or 30 m). We note that this represents a conservative estimate for the neglect of the term, which may in fact have an insignificant effect for even smaller ranges. Also, a similar expansion of Eq. (35) and a comparison of terms neglected with those retained could be carried out. This process yields Eqs. (37) and (38) along with other conditions on slowness of $n(r,\theta,z)$-variations involving various partial derivatives up to third order of $n(r,\theta,z)$. We omit these conditions for brevity.

To summarize, the farfield approximation has been argued as valid for ranges r bigger than r_f given by Eq. (25). Similarly, the factorization approximation in conjunction with the rational-function approximation of the operator Q is appropriate for slow variations in $n(r,\theta,z)$ [see Eq. (37)] and for r bigger than $\delta^{1/2} r_f$ [see Eq. (39)]. When these conditions hold, the three-dimensional wide angle PE, Eq. (20), should be applied. As range increases such that $\frac{1}{(k_0 r)^2} \frac{\partial^2 u}{\partial \theta^2}$ is negligible, the two-dimensional wide angle PE, Eq. (21), should be used. In the two-dimensional application, if we regard the θ-partitions as N, this coincides exactly with the "N x 2D Problem" defined by Perkins and Baer.[3] It is important to remark that Eqs. (37) through (39) are not predicated on any statement concerning the size of the θ-variation of u. The conditions for validity of both the farfield and the factorization approximations are independent of whether or not the θ-derivatives in Eq. (20) affect the propagation significantly. One resolution of this latter issue is provided by using the scaling arguments presented below to compare the $(k_0 r)^{-2}$ $(\partial^2/\partial\theta^2)$ terms with $\partial^2 u/\partial r^2$. It is sufficient here to remark that azimuthal variation in u can be introduced by three mechanisms: water-column variations; n boundary fluctuations, either from bottom topography and structure or from surface irregularities; and directionality in the representation of the source nearfield.

MULTISCALE FORMALISM

In this section we provide a systematic asymptotic derivation for a class of wide-angle 3D PEs. One advantage is that the farfield, rational-function, and factorization approximations, which were explicitly required in the previous development, are not necessary here. Instead the scaling and asymptotic expansions produce the effects of these approximations without any additional assumptions. Moreover, considerable insight is gained into the nature of these approximations and the conditions for their validity. The wide-angle 3D PE, Eq. (20) is found as an important case under well-defined conditions.

We begin with the Helmholtz equation [Eq. (1)] which does not incorporate source effects. In this paper, we do not treat a scaling and asymptotic expansion appropriate to the near-source region. Consequently, we regard the spatial portion of acoustic pressure as a specified function of depth and azimuth at some near-source radial distance. Further, we assume boundary conditions are specified at the ocean bottom and surface and for the azimuthal region of interest. For brevity we do not write these conditions in the following development, but they are easily incorporated once physical models for the boundaries are specified. The only spatial conditions which we explicitly employ are the obvious ones of bounded pressure for all ranges and of only an outgoing wave at large ranges.

With some choice of the reference sound speed c_0, we assume that $n^2(r, \theta, z)$ can be written in the form

$$n^2(r,\theta,z) = 1 + \epsilon\eta(r^*,\theta^*,z^*) \quad , \tag{40}$$

in which η is a function of order-of-magnitude unity[12] and the scaled variables with asterisks are nondimensional. The quantity η could be taken as the maximum relative deviation of c from c_0 which typically is no more than about 10^{-2} in ocean applications. The scaled variables are defined by

$$r^* = \epsilon\, k_0 r, \quad z^* = \epsilon^{1/2}\, k_0 z, \quad \theta^* = \alpha\, \epsilon^{-1/2}\, \theta \quad . \tag{41}$$

The first two variables of Eq. (41) are chosen following Tappert[1] who also provides a justification for the above definition of ϵ. (Other definitions, such as a channel aspect ratio, may be appropriate for certain propagation conditions, as, for example, in an isospeed channel where no definition is ideal in all circumstances.) The third variable contains an ordering parameter α that serves to account for the rapidity of the azimuthal variation in n^2 (r, θ, z) and, more generally, in the solution. Thus, if α is of order unity with respect to ϵ [denoted by $\alpha = O(1)$], then the dimensional azimuthal derivative $r^{-1}\ \partial p/\partial\theta$ is comparable to $\partial p/\partial z$, so that one or more of the three mechanisms mentioned previously are accounting for substantial azimuthal variations. If on the other hand $\alpha = O(\epsilon^{1/2})$, then $r^{-1}\ \partial p/\partial\theta$ is comparable to $\partial p/\partial r$, and azimuthal variations are relatively smaller. Our development proceeds with α, of order one, and extends to other cases.

Employing Eqs. (40) and (41) in Eq. (1) and dropping the asterisks henceforth (so that (r,θ,z) now represent scaled variables), for $p = p(r,\theta,z)$, we obtain

$$\epsilon^2\left(p_{rr} + \frac{1}{r} p_r\right) + \epsilon\left(p_{zz} + \frac{\alpha^2}{r^2} p_{\theta\theta}\right) + (1 + \epsilon\eta)\, p = 0 \quad , \tag{42}$$

where subscripts denote partial derivatives with respect to the scaled variables. Motivated by Eqs. (2) and (6), we apply the method of multiple scales by seeking a solution of Eq. (42) in the form

$$p(r,\theta,z) = P(\rho,r,\theta,z;\epsilon) \quad , \tag{43}$$

where $\rho = r/\epsilon$. With Eq. (43), Eq. (42) can be written as

$$[P_{\rho\rho} + P] + \epsilon\left[2P_{r\rho} + \frac{1}{r} P_\rho + P_{zz} + \frac{\alpha^2}{r^2} P_{\theta\theta} + \eta P\right] + \epsilon^2\left[P_{rr} + \frac{1}{r} P_r\right] = 0 \; . \tag{44}$$

We note that our results are, in fact, unchanged if (for example) the term $\epsilon r^{-1} P_\rho$ is written as $\rho^{-1} P_\rho$, but the analysis would be more involved. We next assume an asymptotic expansion of P, i.e.,

$$P \sim \sum_{n=0}^{\infty} \delta^n P^{(n)}(\rho,r,\theta,z) \; , \qquad \epsilon \to 0 \quad , \tag{45}$$

and inserting Eq. (45) in Eq. (44) produces a sequence of equations. The first three of which are

$$\mathcal{L} P^{(0)} = P^{(0)}_{\rho\rho} + P^{(0)} = 0, \tag{46}$$

$$\mathcal{K}P^{(1)} = -\left[2P^{(0)}_{r\rho} + \frac{1}{r}P^{(0)}_{\rho} + P^{(0)}_{zz} + \frac{\alpha^2}{r^2}P^{(0)}_{\theta\theta} + \eta P^{(0)}\right]$$

$$= MP^{(0)} \quad , \tag{47}$$

and

$$\mathcal{K}P^{(2)} = MP^{(1)} - \left[P^{(0)}_{rr} + \frac{1}{r}P^{(0)}_{r}\right] \quad . \tag{48}$$

The solution of Eq. (46), satisfying the outgoing-wave condition for large ρ, is

$$P^{(0)} = A^{(0)}(r,\theta,z)\ e^{i\rho} \quad . \tag{49}$$

Using Eq. (49) and solving Eq. (47) with the outgoing-wave condition yields

$$P^{(1)} = A^{(1)}(r,\theta,z)\ e^{i\rho} + \frac{i}{2}\left[2iA^{(0)}_{r} + \frac{i}{r}A^{(0)}\right.$$

$$\left. + A^{(0)}_{zz} + \frac{\alpha^2}{r^2}A^{(0)}_{\theta\theta} + \eta A^{(0)}\right]\rho e^{i\rho} \quad . \tag{50}$$

The solution in Eq. (50) is bounded for all ρ only if the bracketed terms sum to zero. Applying this condition and setting

$$A^{(0)} = U^{(0)}(r,\theta,z)/r^{1/2} \quad , \tag{51}$$

we obtain

$$-2iU^{(0)}_{r} = U^{(0)}_{zz} + \frac{\alpha^2}{r^2}U^{(0)}_{\theta\theta} + \eta U^{(0)} \quad . \tag{52}$$

Equation (52), with $\alpha = 1$ [which can be chosen without loss of generality when $\alpha = O(1)$], is precisely Eq. (9), the fundamental narrow-angle 3D PE of Ref. 1. With the definition

$$p^{(1)} = A^{(1)} e^{i\rho} = U^{(1)}(r,\theta,z)\, e^{i\rho}/r^{1/2} \quad . \tag{53}$$

We insert Eqs. (49), (51), and (53) into Eq. (48) and require $P^{(2)}$ to be bounded for all ρ. It follows as before that $u^{(1)}$ must satisfy

$$-2iU_r^{(1)} = U_{zz}^{(1)} + \frac{\alpha^2}{r^2} U_{\theta\theta}^{(1)} + nU^{(1)} + U_{rr}^{(0)} + \frac{1}{4r^2} U^{(0)} \quad . \tag{54}$$

Our results thus far are summarized by

$$p \sim \frac{e^{i\rho}}{r^{1/2}} [U^{(0)} + \epsilon U^{(1)} + O(\epsilon^2)] , \quad \epsilon \to 0 \quad , \tag{55}$$

where $U^{(0)}$ and $U^{(1)}$ are obtained from Eqs. (52) and (54), respectively, for α of order one. When $\alpha = O(\epsilon^{1/2})$, the second term on the right of Eq. (52) [and Eq. (54)] is absent and is replaced by $r^{-2}\, U_{\theta\theta}^{(0)}$.

Having obtained an analogue of the narrow-angle 3D PE, Eq. (9), we next obtain a wider angle version corresponding to Eq. (20). Differentiating Eq. (52) with respect to r yields

$$U_{rr}^{(0)} = \frac{i}{2}\; U_{zzr}^{(0)} - \frac{2\alpha^2}{r^3} U_{\theta\theta}^{(0)} + \frac{\alpha^2}{r^2} U_{\theta\theta r}^{(0)} + n_r\, U^{(0)} + nU_r^{(0)} \quad . \tag{56}$$

If we define an operator

$$n = \frac{i}{2}\left[\frac{\partial^3}{\partial r \partial^2 z} - \frac{2\alpha^2}{r^3}\frac{\partial^2}{\partial\theta^2} + \frac{\alpha^2}{r^2}\frac{\partial^3}{\partial r \partial\theta^2} + n_r + n\frac{\partial}{\partial r}\right] + \frac{1}{4r^2} \quad , \tag{57}$$

and use Eqs. (56) and (57) in Eq. (54), we find

$$-2iU_r^{(1)} = U_{zz}^{(1)} + \frac{\alpha^2}{r^2} U_{\theta\theta}^{(1)} + nU^{(1)} + nU^{(0)} \quad . \tag{58}$$

Adding Eq. (52) and ϵ times Eq. (58) gives

$$-2iU_r = U_{zz} + \frac{\alpha^2}{r^2} U_{\theta\theta} + nU + \epsilon nU - \epsilon^2 nU^{(1)} \quad , \tag{59}$$

where we have defined

$$U = U^{(0)} + \epsilon U^{(1)} \quad . \tag{60}$$

The last term on the right of Eq. (59) should be dropped, as is consistent without neglect of $O(\epsilon^2)$ terms. With this omission and using Eq. (57), Eq. (59) becomes

$$-2i \left[1 + \frac{\epsilon}{4} \left(\frac{\partial^2}{\partial z^2} + \frac{\alpha^2}{r^2} \frac{\partial^2}{\partial \theta^2} + n \right) \right] U_r =$$

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\alpha^2}{r^2} \frac{\partial^2}{\partial \theta^2} + n \right] U + \epsilon \left[\frac{i}{2} \left(n_r - \frac{2\alpha^2}{r^3} \frac{\partial^2}{\partial \theta^2} \right) + \frac{1}{4r^2} \right] U \quad . \tag{61}$$

The resulting wide-angle 3D PE from our asymptotic derivation is given by Eq. (61) when α is order one. Using Eqs. (40) and (41), it follows that Eq. (61) with $\alpha = 1$ agrees exactly with Eq. (20) but with the additional last three terms on its right side. These three terms are multiplied by ϵ (typically about 10^{-2}) and do not involve a radial derivative of the solution so they are in some sense less significant than the remaining terms in Eq. (61). In fact, they can be identified precisely with contributions that were argued as small in the derivation of Eq. (20). Specifically, the first two of those terms correspond to those neglected via Eqs. (37) and (38) in the factorization approximation. Furthermore, the last term corresponds to those dropped through Eqs. (24) and (25) in the farfield approximation. In this way, it follows that Eq. (20) is an apparently consistent approximation

of physical interest to the class of 3D PEs represented by Eq. (61). We note that the result of the analysis for $\alpha = O(\varepsilon^{1/2})$, i.e., for relatively weak azimuthal variations, may be seen from Eq. (61) by setting $\alpha^2 = \varepsilon$ and dropping the two $O(\varepsilon^2)$ terms. Estimation of the appropriate magnitude for α in any specific application depends on detailed consideration of the three azimuthally-directive mechanisms mentioned previously.

A VALIDITY TEST

The accuracy of the 3D wide angle PE has been examined by Schultz, Lee, and Jackson[5] using an exact solution test. Their exact solution $u(r,\rho,z)$ is required to take the form

$$u(r,\theta,z) = \sin(\Omega z)\, e^{im\theta}\, \phi(r) \quad , \tag{62}$$

where $\phi(r)$ satisfied the differential equation

$$\frac{d\phi}{dr} = \left(\frac{\frac{-i}{2}\, k_o\, m^2/(k_o r)^2}{1 - \frac{1}{4}\, m^2/(k_o r)^2} \right) \phi \quad . \tag{63}$$

For appropriate choices of z = an integer multiple of π, m=1, and a solution of Eq. (63), the expression of (62) satisifes the 3D wide angle PE equation (20). On the other hand, Eq. (20) is solved by an implicit finite difference method that discretizes Eq. (20) into a large sparse system of equations. This system was solved by a Yale sparse technique whose solution compares favorably with the exact solution (62) at every range and at every ρ sector. Results of comparison have been reported in Ref. 5.

We want to establish the validity of Eq. (20) and compare the solution of Eq. (20) with a known reference solution for an application (as reported by Baer[2]). This application problem deals with a profile that can be calculated by the formula $(c(r,\theta,z) = c_m(z) + (0.001)\, r\sin(\theta)$, where $c_m(z)$ takes on the values described by the table below in the vertical plane at 0°.

z (m)	c(z) (m/s)
0.0	1536.5
152.4	1539.243
406.3	1501.143
1015.9	1471.882
5587.91	1549.606
5587.91	1555.526

In the calculation, the source is placed at 254 m below the surface with a frequency of 25 Hz, and the receiver is placed at 815 m. The propagation is carried out up to 140 km in range. Results are produced in azimuthal sectors. The sector boundaries are assumed absorbing. Of particular interest is the result taken from the sector [-20°, 20°] at range 120 km. Reference results were reported by Baer[2] in the same sector using the split-step Fourier algorithm. In figure 1, the solid line is the 3D PE result, the dashed line is the Nx2D result, and the dotted line is the 3D wide angle PE result, which was calculated by the Yale sparse technique. It is seen that the 3D wide angle PE result compares closely with reference solutions.

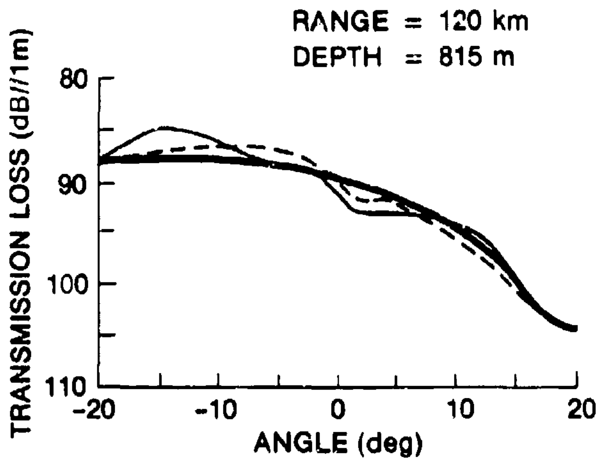

Figure 1. Results from Sector [-20°, 20°] at 120 km

CONCLUSIONS

A wide angle partial differential equation has been developed to predict the underwater sound propagation in three dimensions. This partial differential equation is of the third order in theory. It is named after the 3D wide angle PE because the small 3D PE is a special case. The entire development was based on an operator factorization whose theory was fully justified by the operator analysis and supported by the multiple scale analysis. The most important result is the information to indicate when and where the three-dimensional problem can be solved two-dimensionally. The mathematical validity was established by Schultz, Lee, and Jackson[5] in their numerical solution; however, the simulated example demonstrated further the 3D wide angle PE capability. This 3D Wide Angle PE is, by far, a more general purpose model with useful flexible capabilities.

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3. DERIVATION, CONSISTENCY, AND STABILITY OF AN IMPLICIT FINITE DIFFERENCE SCHEME

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ABSTRACT: Parabolic equation (PE) approximations to the reduced wave equation (Helmholtz equation) are used extensively in the prediction of long-range sound propagation in ocean environments. In two dimensions parabolic approximating partial differential equations have been traditionally solved numerically via a Green's function approach (Fast Field Program) and a Fast Fourier Transform (split-step). Recently, Lee et al. created an implicit finite difference (IFD) program to solve more general two-dimensional PE approximations (those that accommodate wider angles of propagation).

In this paper, we present a three-dimensional PE (encompassing small and wide angles) that is a third order partial differential equation, and derive an IFD scheme to solve it numerically. The numerical scheme is presented in several different ocean environments, a wedge shaped region with absorbent bottom and sides, the same region with hard bottom, and a full 360° propagating region with soft/hard bottom. Matrix formulations are carefully worked out in anticipation of the implementation. We derive the consistency of the scheme with the original partial differential equation and show that the scheme is second order accurate. Finally, we present a discussion of the stability properties that might be exhibited by the scheme.

INTRODUCTION

In this section we shall discuss the derivation of the implicit finite difference scheme associated with the wide angle three-dimensional parabolic approximation. We shall prove that the difference equation is unconditionally consistent with the partial differential equation and investigate the stability of the scheme. The finite difference approximation is a Crank-Nicolson type scheme. We shall show that it has consistency properties that are very much like those of the classical Crank-Nicolson scheme when applied to the canonical heat equation. In this regard we remark that the straight forward explicit difference scheme is stable under certain conditions on the parameters when applied to the heat equation, but is unstable for all combinations of the parameters even when applied to the simplest of our two dimensional parabolic partial differential equations.

A CRANK-NICOLSON TYPE APPROXIMATION SCHEME

The wide angle approximating parabolic equation (PE) is given by

$$(1 + q_1 L_1 + q_2 L_2)\frac{\partial u}{\partial r} = ik_0((p_1 - q_1) L_1 + (p_2 - q_2) L_2)u,$$

$$L_1 u = \left[(n^2(r,z,\theta) - 1) + (1/k_0^2)(\partial^2/\partial z^2)\right]u \ , \ L_2 u = \left[(1/k_0^2 r^2)(\partial^2/\partial\theta^2)\right]u \qquad (1)$$

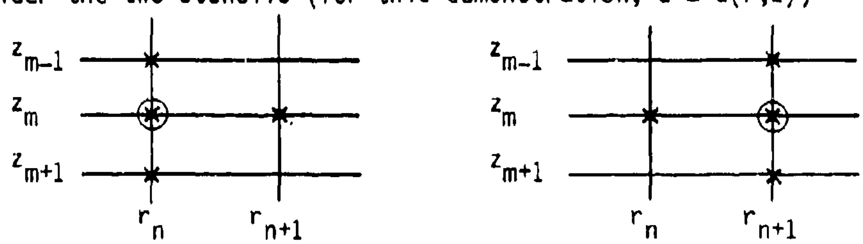
$u = u(r,z,\theta)$, $p_1 \neq q_1$, $p_2 \neq q_2$. We shall let $u_m^n = u(r_n, z_m, \theta\ )$, where $r_n = r_0 + nk$, $r_0 > 0$, $z_m = mh$, $\theta_\ell = \theta_0 + \ell d$, thus $\Delta r = k$, $\Delta z = h$, $\Delta\theta = d$, the

limits of the indexes are $m = 0,1,\ldots, M$ or $M + 1$, $\ell = 0,1,\ldots, L$ or $L + 1$, $n = 0,1,\ldots$. In an abuse of the notation, we shall use n in two different ways: as the index of refraction and as a counter of the number of range steps. (The context should make it clear which is intended in each case.) We wish to derive a Crank-Nicolson type approximation to (1).

A standard way in which the Crank-Nicolson approximation is derived for traditional parabolic partial differential equations is to take the average of the classic explicit (forward) difference approximation and the implicit (backward) approximation. In order to motivate the application of this procedure to (1), we shall briefly describe its application to a PE in standard form, namely, the small angle Tappert equation

$$u_r = cu + du_{zz}\ , \quad c = ik_0(n_0^2 - 1)/2, \quad d = i/2k_0 \quad . \tag{2}$$

Consider the two stencils (for this demonstration, $u = u(r,z)$)



The first of these is used to make the forward approximation based at the point (r_n,z_m); the second is used to make the backward approximation based at (r_{n+1},z_m). The difference equations are

$$(u_m^{n+1} - u_m^n)/k = c_m^n\, u_m^n + d(u_{m+1}^n - 2u_m^n + u_{m-1}^n)/h^2 \quad , \tag{3a}$$

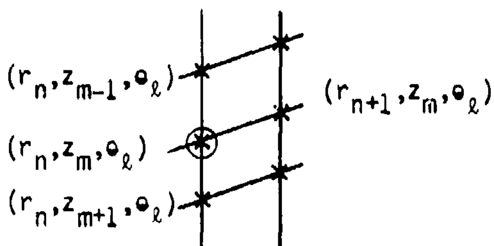
and

$$(u_m^{n+1} - u_m^n)/k = c_m^{n+1}\, u_m^{n+1} + d[u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}]/h^2 \quad . \tag{3b}$$

Note that the left-hand sides of these equations are the same. The Crank-Nicolson approximation to (2) is obtained in taking ((3a) + (3b))/2.

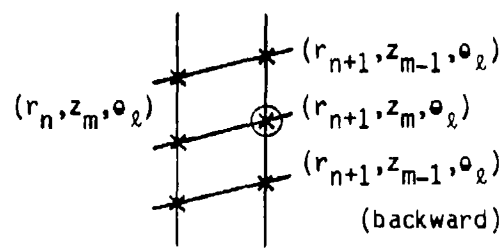
In order to begin to carry out this development for (1) we need to define the forward and backward (in r) discretizations of $\frac{\partial^2}{\partial z^2} \frac{\partial u}{\partial r}$ and $\frac{\partial^2}{\partial \theta^2} (\frac{\partial u}{\partial r})$. In each case we shall take the centered difference in the second order variable and the standard difference in r. The key to taking the forward and backward differences in r is to keep the base point of the stencils, the point at which the approximations are being made, clearly in mind.

The two stencils associated with the z-derivative, base point encircled are



(forward)

,



(backward) .

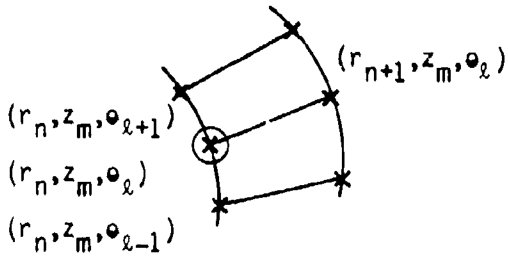
The two difference approximations to u_{rzz} are equal, as above (the forward and backward approximations to the full differential equation (1) are not equal though), and have the common value

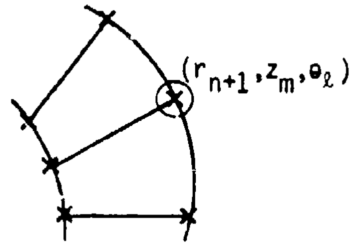
$$\left[(u^{n+1}_{m+1,\ell} - 2u^{n+1}_{m,\ell} + u^{n+1}_{m-1,\ell})/h^2 - (u^n_{m+1,\ell} - 2u^n_{m,\ell} + u^n_{m-1,\ell})/h^2\right]/k \quad .$$

Henceforth we shall use the central difference operator notation

$$(\delta^2_m u)^n_\ell = (u^n_{m+1,\ell} - 2u^n_{m,\ell} + u^n_{m-1,\ell}) \quad .$$

A completely analogous development takes place for the second derivative in θ. The corresponding stencils in the r,θ directions are as follows:





The approximations to $u_{r\theta\theta}$ have the common value

$$\left[(\delta^2 u)^{n+1}_m \Big/ d^2 - (\delta^2_\ell u)^n_m \Big/ d^2\right] \Big/ k \quad ,$$

where

$$(\delta^2_\ell u)^n_m = u^n_{m,\ell+1} - 2u^n_{m,\ell} + u^n_{m,\ell-1} \quad .$$

It is not difficult to prove that for arbitrary sufficiently differentiable functions $\phi(r,z,\theta)$,

$$\begin{aligned}(\phi_{rzz})^n_{m,\ell} &- \left[(\delta^2_m u)^{n+1} - (\delta^2_m u)^n_\ell\right] \Big/ kh^2 \\ &= -(\phi_{rrzz})^n_{m,\ell}\,(k/2) - (\phi_{r4z})^n_{m,\ell}\,(h^2/12) \qquad (4) \\ &\quad -(\phi_{rr4z})^n_{m,\ell}\,(kh^2/24) + O(k^2 + n^2)\end{aligned}$$

as $h \to 0$, $k \to 0$ independently of the manner in which h,k approach zero. A completely analogous formula holds for $\phi_{r\theta\theta}$, which we shall use in the ensuing development.

We shall now obtain the desired difference scheme by producing the analogue of (3a) and (3b) for equation (1). For the first of these, the base point is (r_n,z_m,θ_ℓ) and, thus, the forward approximation is given by

$$(1 + q_1((n^2)^n_{m,\ell} - 1)\ (u^{n+1}_{m,\ell} - u^n_{m,\ell})/k) + q_1(1/k_o^2)\left[(\delta_m^2 u)^{n+1}_\ell - (\delta_m^2 u)^n_\ell\right]/kh^2$$

$$+ q_2(1/(k_o r_n)^2 \left[(\delta_\ell^2 u)^{n+1}_m - (\delta_\ell^2 u)^n_m\right]/kd^2) \qquad (5a)$$

$$= ik_o(p_1 - q_1)((n^2)^n_{m,\ell} - 1)u^n_{m,\ell} + (i/k_o)(p_1 - q_1)(\delta_m^2 u)^n_\ell/h^2$$

$$+(i/\kappa_o r_n^2)(p_2 - q_2)(\delta_\ell^2 u)^n_m\ /d^2 \quad .$$

The backward approximation has base point $(r_{n+1}, z_m, \theta_\ell)$ and is given by

$$(1 + q_1((n^2)^{n+1}_{m,\ell} - 1)\ (u^{n+1}_{m,\ell} - u^n_{m,\ell})/k) + q_1(1/k_o^2)\ (\delta_m^2 u)^{n+1}_\ell - (\delta_m^2 u)^n_\ell\ /kh^2$$

$$+ q_2(1/(\kappa_o r^2_{n+1})\left[(\delta_\ell^2 u)^{n+1}_m - (\delta_\ell^2 u)^n_m\ /kd^2)\right] \qquad (5b)$$

$$= ik_o(p_1 - q_1)((n^2)^{n+1}_{m,\ell} - 1)u^{n+1}_{m,\ell} + (i/k_o)(p_1 - q_1)(\delta_m^2 u)^{n+1}_\ell\ /h^2$$

$$+(i/k_o r^2_{n+1})(p_2 - q_2)(\delta_\ell^2 u)^{n+1}_m\ /d^2 \quad .$$

Finally, the average of these two yields the Crank-Nicolson difference approximation system. After considerable simplification the system can be expressed as follows:

$$(\overline{b}/h^2)u^{n+1}_{m-1,\ell} + (\overline{b1}^n/d^2)u^{n+1}_{m,\ell-1} + (\overline{a1}^n_{m,\ell} - (2/h^2)\overline{b}$$

$$- (2/d^2)\overline{b1}^n)u^{n+1}_{m,\ell} + (\overline{b}/h^2)u^{n+1}_{m+1,\ell} + (\overline{b1}^n/d^2)u^{n+1}_{m,\ell+1} \qquad (6)$$

$$= (b/h^2)u^n_{m-1,\ell} + (b0^n/d^2)u^n_{m,\ell-1} + (a0^n_{m,\ell} - (2/h^2)b$$

$$-(2/d^2)b0^n)u^n_{m,\ell} + (b/h^2)u^n_{m+1,\ell} + (b0^n/d^2)u^n_{m,\ell+1} \quad ,$$

where

$$b = b(k) = q_1/k_0^2 + ik(p_1 - q_1)/2k_0 \quad ,$$

$$b1 = b1(r;k) = \left[\frac{q_2}{2k_0^2}\left(\frac{1}{r^2} + \frac{1}{(r+k)^2}\right) + i\,\frac{k(p_2 - q_2)}{2k_0(r+k)^2}\right] \quad ,$$

$$b0 = b0(r;k) = \left[\frac{q_2}{2k_0^2}\left(\frac{1}{r^2} + \frac{1}{(r+k)^2}\right) + \frac{ik(p_2 - q_2)}{2k_0\, r^2}\right] \quad , \tag{7}$$

$$a1 = a1(r,z,\theta;k) = (1 + q_1[(n^2(r,z,\theta) + n^2(r+k,z,\theta))/2 - 1])$$
$$+ ikk_0(p_1 - q_1)(n^2(r+k,z,\theta) - 1)/2 \quad ,$$

$$a0 = a0(r,z,\theta;k) = (1 + q_1[(n^2(r,z,\theta) + n^2(r+k,z,\theta))/2 - 1])$$
$$+ ikk_0(p_1 - q_1)(n^2(r,z,\theta) - 1)/2 \quad .$$

The bar over an expression indicates the taking of the complex conjugate. Note also that a1 and a0 are equal if $n(r,z,\theta)$ is independent of the range variable r.

BOUNDARY CONDITIONS AND MATRIX FORMULATION

We wish to express system (6) in a convenient matrix formulation, but the precise form of the coefficient matrices depends on the boundary conditions imposed on the original problem. Throughout the discussion we shall assume the surface boundary conditions $u(r,0,\theta) = 0$. Another standing assumption is the initial condition, namely, for a given function $f(z,\theta)$, $u(r_0,z,\theta) = f(z,\theta)$.

A frequently imposed bottom boundary condition is $u(r,z_{M+1},\theta) = 0$ (in this formulation the bottom is at $z = z_{M+1}$), the general assumption associated with this condition is that an artificially imposed absorbing layer below the ocean floor prevents energy from entering the water column. Similarly we can consider propagation taking place in a cylindrical sector (pie-shaped region) between two azimuthal angles denoted by θ_0 and θ_{L+1} with an absorption region on each vertical side of the sector. Thus in addition to the conditions

$$u(r,z_0,\theta) = 0 \quad , \quad u(r,z_{M+1},\theta) = 0 \quad , \tag{8a}$$

we have

$$u(r,z,\theta_0) = 0 \quad , \quad u(r,z,\theta_{L+1}) = 0 \quad , \tag{8b}$$

for all $r > 0$, z, $\theta_0 \leq \theta \leq \theta_{L+1}$. This is the case considered by Baer and Perkins for small angle PE. Finally, one assumes a given sound profile at a distance from the source

$$u(r_0,z,\theta) = f(z,\theta) \quad . \tag{8c}$$

The system (6) in conjunction with the boundary conditions (8) can be expressed in a particularly nice symmetric block tridiagonal matrix form. Namely

$$\begin{bmatrix} \overline{AI}_1^n & \overline{BI}^n & & & \\ \overline{BI}^n & \overline{AI}_2^n & \overline{BI}^n & & \\ & \ddots & \ddots & \ddots & \\ & & \overline{BI}^n & \overline{AI}_{L-1}^n & \overline{BI}^n \\ & & & \overline{BI}^n & \overline{AI}_L^n \end{bmatrix} \begin{bmatrix} u_{(1)}^{n+1} \\ u_{(2)}^{n+1} \\ \vdots \\ u_{(L-1)}^{n+1} \\ u_{(L)}^{n+1} \end{bmatrix} = \begin{bmatrix} AO_1^n & BO^n & & & \\ BO^n & AO_2^n & BO^n & & \\ & \ddots & \ddots & \ddots & \\ & & BO^n & AO_{L-1}^n & BO^n \\ & & & BO^n & AO_L^n \end{bmatrix} \begin{bmatrix} u_{(1)}^{n} \\ u_{(2)}^{n} \\ \vdots \\ u_{(L-1)}^{n} \\ u_{(L)}^{n} \end{bmatrix} \tag{9}$$

where each block is M x M, the diagonal blocks are tridiagonal matrices and the off diagonal blocks are diagonal matrices with

$$B0^n = \mathrm{diag}\,[b0^n/d^2, \ldots, b0^n/d^2] \quad ,$$
$$B1^n = \mathrm{diag}\,[b1^n/d^2, \ldots, b1^n/d^2] \quad , \qquad (9a)$$

$$A0^n = \begin{bmatrix} \alpha 0^n_{1,\ell} & \beta & & & & \\ \beta & \alpha 0^n_{2,\ell} & \beta & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \beta & \alpha 0^n_{M-1,\ell} & \beta \\ & & & & \beta & \alpha 0^n_{M,\ell} \end{bmatrix} \qquad (9b)$$

$$\beta = b/h^2, \; \alpha 0^n_{m,} = a0^n_{m,} - (2/h^2)b - (2/d^2)b0^n$$

$$A1^n_\ell = \begin{bmatrix} \alpha 1^n_{1,\ell} & \beta & & & & \\ \beta & \alpha 1^n_{2,\ell} & \beta & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \beta & \alpha 1^n_{M-1,\ell} & \beta \\ & & & & \beta & \alpha 1^n_{M,\ell} \end{bmatrix} \qquad (9c)$$

$$\alpha 1^n_{m,\ell} = a1^n_{m,\ell} - (2/h^2)b - (2/d^2)\, b^{\,n} \quad ,$$

and

$$u^n_{(\ell)} = \begin{bmatrix} u^n_{1,\ell} \\ u^n_{2,\ell} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u^n_{M,\ell} \end{bmatrix} \quad . \tag{9d}$$

The system (9) shall be referred to symbolically as

$$\overline{a1}^n \; u^{n+1} = a0^n u^n \quad .$$

We shall now turn to more general boundary conditions and consider a cylindrically shaped region $0 < \theta \leq 2\pi$. Again we shall retain the pressure release top surface boundary condition

$$u(r,z_0,\theta) = 0 \quad , \quad r > 0 \tag{10a}$$

The bottom boundary conditions are artificially located far below the actual bottom of the wave guide, but in the current case we assume that the position $z = z_M$ is the actual interface between ocean floor and water and allow for the possibility of reflection of rays. For given real constants α, β_0, γ the condition is given by

$$\alpha u(r,z_M,\theta) + \beta_0 u_z(r,z_M,\theta) = \gamma \quad , \quad \beta_0 \neq 0 \quad . \tag{10b}$$

The case of general interest is $\alpha = 0$, $\beta_0 = 1$, $\gamma = 0$. Finally, we impose a continuity condition on the motion in the θ variable, namely

$$u(r,z,0) = u(r,z,2\pi) \ , \ u_\theta(r,z,0) = u_\theta(r,z,2\pi) \ , \ r > 0 \quad . \tag{10c}$$

We shall assume that $\Theta_0 = 0$ and $\Theta_{L+1} = 2\pi$.

First consider the discretization of the boundary condition (10b). We shall use a centered difference to approximate the derivative so as to maintain the second order character of the approximations (as shall be seen in the ensuing development, (6) is a second order scheme). We use

$$u_z(r,z_M,\Theta) \simeq [u(r,z_{M+1},\Theta) - u(r,z_{M-1},\Theta)]/2h \quad ,$$

and thus for (10b) we write

$$\alpha u^n_{M,\ell} + \beta_0[u^n_{M+1,\ell} - u^n_{M-1,\ell}]/2h = \gamma \quad , \quad \beta_0 \neq 0 \quad . \tag{11}$$

The term $u^n_{M+1,\ell}$ is ficticious in this context (recall the bottom is at $u^n_{M,\ell}$) and can be expressed in terms of the real unknowns $u^n_{M,\ell}$, $u^n_{M-1,\ell}$ using (11). In order to encompass (11) into the matrix formulation of the problem, set m = M in (6) and make the substitution, from (11),

$$u^n_{M+1,\ell} = u^n_{M-1,\ell} - 2\alpha h u^n_{M,\ell}/\beta_0 + 2h\gamma/\beta_0 \quad , \tag{11a}$$

to obtain

$$\begin{aligned}
2(\overline{b}/h^2)u^{n+1}_{M-1,\ell} &+ (\overline{bI}^n/d^2)u^{n+1}_{M,\ell-1} + (\overline{aI}^{n+1}_{m,\ell} - (2/h^2)\,\overline{b}\,(1 + \alpha h/\beta_0) \\
&- (2/d^2)\,\overline{bI}^n\; u^{n+1}_{M,\ell} + \overline{bI}^n/d^2\; u^{n+1}_{M,\ell+1} \\
&= 2(b/h^2)u^n_{M-1,\ell} + (bO^n/d^2)u^n_{M,\ell-1} + (aO^n_{M,\ell} - (2/h^2)\,b(1 + \alpha h/\beta_0)) \\
&- (2/d^2)\,bO^n)u^n_{M,\ell} + (bO^n/d^2)u^n_{M,\ell+1} + g_{h,k} \quad ,
\end{aligned} \tag{12}$$

$$g_{h,k} = (2\gamma/\beta_0 h)(b - \overline{b} = 2\gamma k i(p_1 - q_1)/\beta_0 h k_0) \quad .$$

Condition (12), compared with (6), forces a change in the last row of each of the diagonal blocks $a1^n$, $a0^n$, $\ell = 1,2,\ldots,L$, of the system (9).

In particular if one should use a bottom reflecting condition (11) in a sector with absorbing vertical sides these row changes plus the addition of the $g_{h,k}$ vector would be the only change in (9). The resulting matrix system, which we choose to denote by

$$\overline{a1}'^n \, u^{n+1} = a0'^n \, u^n + g \quad , \tag{13a}$$

has the same coefficients as (9) except that for the A0n we substitute

$$A0'^n = \begin{bmatrix} \alpha 0^n_{1,\ell} & \beta & & & & \\ \beta & \alpha 0^n_{2,\ell} & \beta & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \beta & \alpha 0^n_{M-1,\ell} & \beta \\ & & & & 2\beta & \alpha 0'^n_{M,\ell} \end{bmatrix} , \quad = 1,2,\ldots,L, \tag{13b}$$

where

$$\alpha 0'^n_{M,\ell} = a0^n_{M,\ell} - (2/h^2)\, b(1 + \alpha h/\beta_0 \; - (2/d^2)\, b0^n \quad .$$

The analogous change in constructing the $A1'^n$ from (12) is made,

$$\alpha 1'^n_{M,\ell} = a1^n_{M,\ell} - (2/h^2)\, b(1 + \alpha h/\beta_0) - (2/d^2)\, b1^n \quad ,$$

and g is the M x L - column vector

$$g = \left[[0,\ldots,0,g_{h,k}],\ \ldots,\ [0,\ldots,0,g_{h,k}]\right]^T \quad . \tag{13c}$$

The system (13), which does not exhibit the symmetric character of (9), will be shown to be equivalent to a symmetric system.

Returning to the main consideration of this subsection, namely, a cylindrically shaped region with bottom reflecting condition (11), the conditions in (10c) can be represented as follows:

$$u^n_{m,0} = u^n_{m,L+1} \text{ , } [u^n_{m,1} - u^n_{m,0}]/d = [u^n_{m,L+1} - u^n_{m,L}]/d \quad . \tag{14}$$

The dummy index 0 = L+1 is only used to help indicate direction of approach to the vertical plane $\theta = 0$, i.e., $u^n_{m,L+1} = u^n_{m,0}$. The two relations in (14) reduce to

$$u^n_{m,0} = (u^n_{m,1} + u^n_{m,L})/2 \quad . \tag{15}$$

One now re-examines (6) in the critical cases $\ell = 1$, $\ell = L$; $m = 1,2,\ldots,M$. The new system of equations differs from (13a) only in the first and last rows of blocks. It is an (ML) dimensional square system block-tridiagonal-like in form, except for the first and last rows of blocks each of which has one additional block; i.e.,

$$\overline{AI}'^n \, u^{n+1} = AO''^n \, u^n + g \quad , \tag{16}$$

where

$$\overline{AK}_{\ell}^{''n} = \begin{bmatrix} AK_1^{''n} & BK^n & & & (1/2)BK^n \\ BK^n & AK_2^{''n} & B^n & & \\ \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & BK^n & AK_{L-1}^{''n} & BK^n \\ (1/2)BK^n & & & BK^n & AK_L^{''n} \end{bmatrix} \quad K = 0,1$$

$B0^n$, $B1^n$, β are as in (9),

$$AK_{\ell}^{''n} = \begin{bmatrix} \alpha K_{1,\ell}^{''n} & \beta & & & \\ \beta & \alpha K_{2,\ell}^{''n} & \beta & & \\ \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & \beta & \alpha K_{M-1,\ell}^{''n} & \beta \\ & & & 2\beta & \alpha K_{M,\ell}^{''n} \end{bmatrix}, \quad \begin{array}{l} K = 0,1 \\ \ell = 1,2,\ldots,L \end{array} \qquad (16a)$$

where, for

$$\ell = 1,L \; ; \; \alpha K_{m,\ell}^{''n} = aK_{m,\ell}^{n} - (2/h^2)\, b - (3/2d^2)\, bk^n \;, \; m = 1,2,\ldots,M-1 \;;$$

$$\alpha K_{M,\ell}^{''n} = aK_{M,\ell}^{n} - (2/h^2)\, b(1 + \alpha h/\beta_0) - (3/2d^2)\, bK^n \quad ,$$

and for

$$= 2,\ldots,L-1 \;;\; \alpha K_{m,\ell}^{''n} = \alpha K_{m,\ell}^{n} = aK_{m,\ell}^{n} - (2/h^2)\, b - (2/d^2)\, bK^n \;,\; m = 1,2,\ldots,M-1 \;;$$

$$\alpha K_{M,\ell}^{''n} = \alpha K_{M,\ell}^{'n} = aK_{M,\ell}^{'n} - (2/h^2)\, b(1 + \alpha h/\beta_0) - (2/d^2)\, bk^n \quad ,$$

K = 0,1. Of course since everything on the right-hand side of (16) is known, one could write it (but not the left-hand side) without the addition of the corner blocks, using (15) directly and making the corresponding changes in the vector g.

CONSISTENCY OF IMPLICIT FINITE DIFFERENCE (IFD)

A difference equation approximation to a partial differential equation is said to be (unconditionally) consistent with the differential equation if the difference equation approaches the differential equation as the mesh size approaches zero, independently of the manner in which the mesh size approaches zero. More precisely, (6) is consistent with (1) if

$$\Bigg| \frac{1}{k}\Bigg[\frac{\bar{b}}{h^2}\,\phi^{n+1}_{m-1,\ell} + \frac{\overline{b1}^{\,n}}{d^2}\,\phi^{n+1}_{m,\ell-1} + \Big(\overline{a1}^{\,n}_{m,\ell} - \frac{2}{h^2}\,\bar{b} - \frac{2}{d^2}\,\overline{b1}^{\,n}\Big)\,\phi^{n+1}_{m,\ell}$$

$$+ \frac{\bar{b}}{h^2}\,\phi^{n+1}_{m+1,\ell} + \frac{\overline{b1}^{\,n}}{d^2}\,\phi^{n+1}_{m,\ell+1} - \frac{b}{h^2}\,\phi^{n}_{m-1,\ell} - \frac{b0^n}{d^2}\,\phi^{n}_{m,\ell-1} - \Big(a0^n_{m,\ell} \qquad (17)$$

$$- \frac{2}{h^2}\,b - \frac{2}{h^2}\,b0^n\Big)\,\phi^{n}_{m,\ell} - \frac{b}{h^2}\,\phi^{n}_{m+1,\ell} - \frac{b0^n}{d^2}\,\phi^{n}_{m,\ell+1}$$

$$- \Big[(1 + q_1 L_1 + q_2 L_2)\,\phi_r - ik_0\,((p_1 - q_1)L_1 + (p_2 - q_2)L_2\,\phi\Big]^n_m \Bigg|$$

approaches zero as h, k $\rightarrow$ 0, independently of the manner in which h, k approach zero for arbitrary net functions $\phi(r,z,\theta)$ having sufficient differentiability. The factor 1/k is present since in the derivation of (6) we previously cleared the k from the denominator. In order to help simplify (17) we shall express, a0, a1, b0, b1, and b in terms of their consistent parts. Let

$$Ra = Ra(r,z,\theta;k) = 1 + q_1 \left[(n^2(r,a,\theta) + n^2(r + k,z,\theta))/2 - 1\right] \quad ,$$

$$Ia = Ia(r,z,\theta) = ik_o\ (p_1 - q_1)(n^2(r,z,\theta) - 1)/2 \quad ,$$

$$RbN = RbN(r;k) = q_2((1/r^2) + (1/(r + k)^2))/2k_o^2 \quad ,$$

$$IbN = IbN(r) = i(p_2 - q_2)/2k_o r^2 \quad ,$$

$$Rb = q_1/k_o^2 \quad , \qquad Ib = i(p_1 - q_1\ /2k_o \quad ,$$

$$c = c(r,z,\theta;k) = q_1[n^2(r + k,z,\theta) - n^2(r,z,\theta)]/2 \quad ,$$

then

$$a0(r,z,\theta;k) = Ra(r,z,\theta;k) + kIa(r,z,\theta) \quad ,$$

$$a1(r,z,\theta;k) = Ra(r,z,\theta;k) + kIa(r+k,z,\theta) \quad ,$$

$$b0(r;k) = RbN(r;k) + kIbN(r) \quad ,$$

$$b1(r;k) = RbN(r;k) + kIbN(r + k) \quad ,$$

and

$$b = RB + kIb \quad .$$

It follows that

$$(1 + q_1\ L_1)\ \phi_r = (Ra - c)\ \phi_r + Rb\phi_{rzz} \quad ,$$

$$ik_o(p_1 - q_1)\ L_1\phi = 2(Ia\phi + Ib\phi_{zz}) \quad ,$$

$$ik_o\ (p_2 - q_2)\ L_2\phi = 2IbN\phi_{\theta\theta} \quad ,$$

and that the standard Taylor approximation in the r variable yields

$$c(r,z,\theta;k) = q_1 n_r^2(r,z,\theta)\ k/2 + O(k^2) \quad ,$$

and

$$RbN(r;k) = q_2/k_o^2 r^2 - kq_2/k_o^2 r^3 + O(k^2) \quad .$$

Now (17) can be expressed in the form

$$\Bigg| \; Ra^n_{m,j}\left[\frac{\phi^{n+1}_{m,j}-\phi^n_{m,j}}{k}\right]-(\phi_r)^n_{m,j} + (Rb - kIb)\left[\frac{(\delta^2_m\phi)^{n+1}_j-(\delta^2_m\phi)^n_j}{kh^2}\right]$$

$$+ (RbN^n - kIbN^n)\,\frac{(\delta^2_j\phi)^{n+1}_m-(\delta^2_j\phi)^n_m}{kd^2} + \left[c^n_{m,j}\right]\,(\phi_r)^n_{m,j}$$

$$- Ia^n_{m,j}\left[\phi^{n+1}_{m,j}-\phi^n_{m,j}\right] - 2Ib\left[\frac{(\delta^2_m\phi)^n_j}{h^2}\right] - \left[\phi^{n+1}_{m,j}\right]\left[Ia^{n+1}_{m,j}-Ia^n_{m,j}\right]$$

$$- \left[IbN^{n+1} - IbN^n\right]\left[\frac{(\delta^2_j\phi)^{n+1}_m}{d^2}\right] - 2IbN^n\left[\frac{(\delta^2_j\phi)^n_m}{d^2}\right] - Rb(\phi_{rzz})^n_{m,j}$$

$$-q_2\,\frac{1}{k_o^2r_n^2}\,(\phi_{r\theta\theta})^n_{m,j} + 2Ib(\phi_{zz})^n_{m,j} + 2IbN^n\,(\phi_{\theta\theta})^n_{m,j} \; \Bigg| \quad .$$

Each term appearing in brackets, [...], can be expanded using a standard Taylor approximation, the centered difference approximation, or (4), thus we obtain

$$Ra^n_{m,j}\left[(\phi_{rr})^n_{m,j}\,k/2 + O(k^2)\right] + (Rb - kIb)\left[(\phi_{rzz})^n_{m,j} + (\phi_{rrzz})^n_{m,j}\,k/2\right.$$

$$\left. + O(h^2 + k^2)\right] + (RbN^n - kIbN^n)\left[(\phi_{r\theta\theta})^n_{m,j} + (\phi_{rr\theta\theta})^n_{m,j}\,k/2 + O(d^2 + k^2)\right]$$

$$+ \left[q_1\,(n^2_r)^n_{m,j}\,k/2 + O(k^2)\right](\phi_r)^n_{m,j} - Ia^n_{m,j}\left[(\phi_r)^n_{m,j}\,k + O(k^2)\right]$$

$$- 2Ib\left[(\phi_{zz})^n_{m,j} + O(h^2)\right] - \left[\phi^n_{m,j} + O(k)\right]\left[(Ia_r)^n_{m,j}\,k + O(k^2)\right]$$

$$-\left[(IbN_r)^n_{m,j}\,k + O(k^2)\quad(\phi_{\theta\theta})^n_{m,j} + O(k) + O(d^2)\right] - 2IbN^n\left[(\phi_{\theta\theta})^n_{m,j} + O(d^2)\right]$$

$$-Rb(\phi_{rzz})^n_{m,j} - q_2 \frac{1}{k_o^2 r_n^2} (\phi_{r\theta\theta})^n_{m,j} + 2Ib(\phi_{zz})^n_{m,j} + 2IbN\ (\phi_{\theta\theta})^n_{m,j}$$

$$= \Big| (k/2) \Big[Ra\phi_{rr} + Rb\phi_{rrzz} + RbN\phi_{rr\theta\theta} + q_1(n^2)_r\ \phi_r - (2q_2/k_o^2 r^3)\ \phi_{r\theta\theta} - 2\ (Ia\phi_r + Ib\ \phi_{rzz}) - 2IbN\ \phi_{r\theta\theta} - 2Ia_r\phi - 2IbN_r(\phi_{\theta\theta})^n_{m,j} \Big] + O(h^2 + k^2 + d^2) \Big|$$

$$= \Big| (k/2)\ \partial\Big[(1 + q_1\ L_1 + q_2\ L_2)\phi_r - ik_o\ ((p_1 - q_1)\ _1 + (p_2 - q_2)L_2)\ \phi\Big]/\partial r^n_{m,j} + O(h^2 + k^2 + d^2) \Big| \quad , \qquad (18)$$

where the last equality uses the fact that $kc\phi_{rr} = O(k^2)$. It follows immediately from the equality of (17) and (18) that in the range dependent index of refraction case, $n = n(r,z,\theta)$, the Crank-Nicolson difference scheme (6) is unconditionally consistent with the partial differential equation (1).

Further, the truncation error or local discretization error can be obtained as the magnitude of the difference, at a point $(r_n, z_m, \theta\ )$ between the differential equation and the difference equation both evaluated with the net function $\phi = u$, the exact solution of the partial differential equation. Again, the equality of (17) and (18) yields immediately the result that the local discretization error of (6) is $O(h^2 + k^2 + d^2)$.

TRANSFORMATION AND NONSINGULARITY OF THE DIFFERENCE SYSTEM

The system (16) is a square system of equations having M x L unknowns. We shall show that a solution always exists, i.e., that $al^{"n}$ is nonsingular, under practical assumptions on the parameters in (1). Looking at (16a) one

can see that the last row of each $AK_j''^n$ causes nonsymmetry to enter (16). We shall show that a very simple transformation of the original system replaces it with an equivalent system in which the $AK_j''^n$ are symmetric, K = 0,1.

First, we shall rewrite (16) in a manner which brings out the structure of the individual blocks $AK_j''^n$. Let DK_j^n be the M x M diagonal matrices having diagonal elements as follows:

$$DK_j^n = \text{diag}\left[aK_{1,j}^n,\ aK_{2,j}^n,\ \ldots,\ aK_{M,j}^n\right],\ j = 1,\ldots,L\ ;\ K = 0,1,$$

and T be the M x M tridiagonal matrix

$$T = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2(1 + \frac{\alpha h}{\beta_0}) \end{bmatrix},$$

then for

$$j = 1,\ L,\ AK_j''^n = DK_j^n - (b/h^2)T - (bK^n/d^2)\ (3/2)I,$$

and for

$$j = 2,\ \ldots,\ L - 1,\ AK_j''^n = DK_j^n - (b/h^2)T - (bK^n/d^2)(2I),\qquad K = 0,1,$$

where I is the M x M identity matrix. Let DK^n and J denote the ML x ML (block) diagonal matrices

$$DK^n = \text{diag}\ [DK_1^n,\ DK_2^n,\ \ldots,\ DK_L],\qquad K = 0,1,$$

$$J = \text{diag}\ [I,\ I,\ \ldots,\ I]$$

(J is the ML x ML identity matrix), and let S be the block tridiagonal matrix with two additional blocks

$$S = \begin{bmatrix} (\frac{3}{2})I & -I & & & (\frac{1}{2})I \\ -I & 2I & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & 2I & -I \\ (\frac{1}{2})I & & & -I & (\frac{3}{2}) \end{bmatrix} ,$$

then

$$AK''^{n} = DK^{n} - (b/h^{2})TJ - (DK^{n}/d^{2})I\ S \quad , \quad K = 0,1 \quad ,$$

where we are using some obvious scalar block multiplication of M x M matrices and ML x ML matrices.

We shall now pursue the transformations alluded to above. The nonsymmetry obviously arises from the involvement of the matrix T. Let P be the M x M diagonal matrix, i.e.,

$$P = \mathrm{diag}\ [1, \ldots, 1, 1/\sqrt{2}] \quad ,$$

then $T = P^{-1}SP$, where S is the M x M symmetric tridiagonal matrix having exactly the same entries as T except in the lower right 2 x 2 block where S is of the form

$$S(\text{lower block}) = \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2(1 + \alpha h/\beta_0) \end{bmatrix} .$$

Let P, S_0 be the ML x ML block diagonal matrices

$$P = \mathrm{diag}\,[P, \ldots, P] \quad , \quad S_0 = \mathrm{diag}\,[S, \ldots, S] \quad ,$$

then

$$AK''^n = DK^n - (b/h^2)^{-1} SPD - (bK^n/d^2) IS$$

$$= P^{-1} DK^n - (b/h^2) P^{-1} S_0 P - (bK^n/d^2) P^{-1} SP$$

$$= P^{-1} [DK^n - (b/h^2) S_0 - (bK^n/d^2) S] P$$

$$\equiv P^{-1} A_t K''^n P \quad , \quad K = 0,1 \quad .$$

Thus the system (16) may be expressed in the symmetric form

$$\overline{A_t 1}''^n v^{n+1} = A_t 0''^n v^n + Pg \quad , \qquad (19)$$

where

$$u^n = P^{-1} v^n \quad ,$$

and

$$A_t K''^n = DK^n - (b/h^2) SJ - (bK^n/d^2) IS \quad , \quad K = 0,1 \quad . \qquad (19a)$$

In order to obtain the nonsingularity of the system we need to derive conditions under which the matrix in (19a) is nonsingular. Let $A = A_t 1''^n$, n be fixed but arbitrary, and suppose there exists an ML-vector γ, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_L)$, $\gamma_1, \ldots, \gamma_L$ M vectors, such that $\gamma = 0$, then $\gamma^* \gamma = 0$, i.e.,

$$(b/h^2)\, \gamma^*{}_0 \gamma + (b1^n/d^2)\, \gamma^* S \gamma = \gamma^* D 1^n \gamma \quad .$$

The quadratic terms $\gamma^* S_0 \gamma, S\gamma^* \gamma$ are real since S_0, S are real symmetric and it is easy to see that $\gamma^* S \gamma \geq 0$. Now separate the equation into two equations by taking its real part and its imaginary part, then eliminate $\gamma^* S_0 \gamma$ from the two equations. The result is the expression

$$\left[\frac{1}{2}\left(\frac{1}{r_n^2}+\frac{1}{r_{n+1}^2}\right)\frac{q_2(p_1-q_1)}{k_o^2d^2}-\left(\frac{1}{r_{n+1}^2}\right)\frac{q_1(p_2-q_2)}{k_o^2d^2}\right]\gamma^*S\gamma$$

$$=(p_1-q_1)\sum_{j=1}^{L}\sum_{m=1}^{M}\left(1+q_1\left((n^2)_{m,j}^{n}-(n^2)_{m,j}^{n+1}\right)/2\right)\left|\gamma_j^{(m)}\right|^2 \quad , (20)$$

$\gamma_j = (\gamma_j^{(1)}, \ldots, \gamma_j^{(M)})$. We wish to state conditions under which it is impossible for (20) to hold except for γ the zero vector. The first condition we shall state is where the index of refraction is slowly varying in range. This is a standard assumption which is frequently utilized long before this point in a development such as this in the general area of underwater acoustics. We implement the condition here to imply that the difference involving n^2 in (20) is small. The standard choices of the p,q parameters are $p_1 = p_2 = 3/4$, $q_1 = q_2 = 1/4$; $p_1 = p_2 = 1/2$, $q_1 = q_2 = 0$, or values close to these. Under such circumstances, the right side of (20) can be seen to be close to the magnitude of the original vector γ , which can be taken to be unity (if $\gamma \neq 0$), and the left side of (20) can be made arbitrarily close to zero by choosing appropriate range step sizes k. Thus, we conclude that under appropriate conditions on the parameters that γ must be the zero vector, i.e., the difference system is nonsingular.

STABILITY

We now turn to the question of the stability of the scheme. A difference system is said to be stable if an error (initial, round-off, etc.) made at the nth step does not magnify uncontrolled in its propagation to the (n + 1)th. In the simplest cases this translates into a need to show that for a system of the form $u^{n+1} = Bu^n + g$, the eigenvalues of B are less than or equal to unity is magnitude.

In the current development the complicated nature of the matrix B^n, from (19), is

$$B^n = (\overline{A_t 1}^{\,n})^{-1} \quad (A_t 0^{n}) \quad ,$$

which makes it very difficult to attack the question of the magnitude of the eigenvalues of B^n. Thus we shall pursue a more heuristic discussion of the stability question. To simplify matters somewhat more we shall consider only the small angle PE, i.e., $p_1 = p_2 = 1/2$, $q_1 = q_2 = 0$, and we shall assume the index of refraction is constant, i.e., $n(r,\theta,z) = n$. Thus the partial differential equation becomes

$$u_r = ik_0 \, p_1 \left[(n^2 - 1) \, u + \frac{1}{k_0^2} \frac{\partial^2 u}{\partial z^2} \right] + \frac{p_2}{k_0^2 r^2} \frac{\partial^2 u}{\partial \theta^2} \quad . \tag{21}$$

The von Neumann or Fourier series method of analyzing stability is a method that actually applies only to linear difference equations with constant coefficients and then only to initial value problems with periodic initial data. In practice the method is widely used outside of this narrow band of problems and it frequently gives useful results.

We begin the method upon assuming that a solution of the difference equation (6) is given by

$$u^n_{m,} = e^{i\omega(mh)} \, e^{i\nu(jd)} \, \xi^n \quad , \tag{22}$$

where $\xi = e^{\alpha k}$, α a complex constant. We seek conditions under which (22) satisfies (6) and $|\xi| \leq 1$ for all n. Frequently ξ is called the amplification. Substituting (22) into (6) and simplifying one obtains

$$\xi \left[-\bar{b} \left[\frac{2}{h^2} (1 - \cos \omega h) + \frac{2}{d^2 r_{n+1}^2} (1 - \cos \nu h) - k_o^2 (n^2 - 1) \right] + 1 \right]$$

$$= \left[-b \left[\frac{2}{h^2} (1 - \cos \omega h) + \frac{2}{d^2 r_n^2} (1 - \cos \nu h) - k_o^2 (n^2 - 1) \right] + 1 \right] , \tag{23}$$

where we have used the fact that $b1 = b/(r + k)^2$, $b0 = b/r^2$, and $a1 = a0 = 1 + bk_o^2(n^2 - 1)$ due to the assumptions enumerated above.

Case 1:

If the index of refraction $n = 1$ then

$$|\xi| = \frac{1 + \left(\frac{k}{4k_o}\right)^2 \left[\frac{2}{h^2} (1 - \cos \omega h) + \frac{2}{d^2 r_n^2} (1 - \cos \nu h) \right]^2}{1 + \left(\frac{k}{4k_o}\right)^2 \left[\frac{2}{h^2} (1 - \cos \omega h) + \frac{2}{d^2 r_{n+1}^2} (1 - \cos \nu h) \right]^2} . \tag{24}$$

We do not mean to imply that there is no dependence on the index n in the left side of (24). We are here concerned only with the dependence of ξ on n which might be of magnitude greater than unity. Indeed separation of variables in (21) indicates that its solution as a function of r has a factor of the form

$$R(r) = e^{ik_o p_1 (n^2 - 1) r} e^{-ur} e^{\lambda / r} ,$$

u,λ constants. It is apparent from (24) that $|\xi| > 1$ which suggests instability of the scheme in this simplest of cases. Extensive numerical computations have not yet been carried out in connection with the scheme. If it should turn out that the computations indicate no instability for this particular case then it will be necessary to abandon the Fourier method and analyze stability via a different approach.

Case 2:

If $n > 1$ and $k_o(n^2 - 1)$ is such that

$$X_n = [\frac{2}{n^2}(1 - \cos \omega h) + \frac{2}{d^2 r_n^2}(1 - \cos \nu h) - k_o^2 (n^2 - 1)] < 0 \quad ,$$

then

$X_{n+1} < X_n < 0$, and thus

$$|\xi| = \frac{1 + (\frac{k}{4k_o})^2 X_n^2}{1 + (\frac{k}{4k_o})^2 X_{n+1}^2} < 1 \quad .$$

Hence the method indicates stability.

4. APPLICATION OF THE YALE SPARSE TECHNIQUE TO SOLVE THE THREE-DIMENSIONAL PARABOLIC EQUATION

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ABSTRACT: The Yale University sparse matrix technique is an efficient method for solving large sparse systems of linear equations such as those that arise at each step in the numerical integration of the stiff system of ordinary differential equations resulting from the application of the finite difference discretization to the three-dimensional parabolic wave equation. We discuss the procedure of a special technique, the Conjugate Gradient method for Normal Equations (CGNE) together with its advantage for solving three-dimensional underwater acoustic wave propagations.

INTRODUCTION

Interest in applying the parabolic equation (PE) approximation to solve three-dimensional ocean acoustic wave propagation is on the rise. A three-dimensional parabolic equation (3D PE), originally introduced by Tappert,[1] dealing with small angle propagation was solved by Baer-Perkins[2] effectively. Baer-Perkins solved the 3D PE by means of the split-step algorithm extended to three-dimensional calculations. Their efficiency in calculation is to specialize the problem into N by two-dimensional problems (N x 2D algorithm). A second three-dimensional wave equation was recently developed by Siegmann, Lee, and Kriegsmann[3] that offers the wide angle capability (3D wide angle PE). Reference 3 showed that the 3D PE is a special case of the 3D wide angle PE. Since the 3D PE is a special case of the 3D wide angle PE, we proceed only to seek the solution of the 3D wide angle PE. Bayliss, Goldstein, and Turkel[4] used the sparse matrix technique and effective preconditioning to solve the Helmholtz equation. For the solution of our problem, we introduce the Yale University sparse matrix technique. A brief discussion on the Yale sparse technique will be given in the next section. In order to set up the 3D wide angle PE in the form solvable by the Yale sparse technique, we apply an implicit finite difference scheme to formulate the 3D wide angle PE into a finite difference equation. Numerical solution to this implicit finite difference equation is carried out by the convergent Crank-Nicolson scheme. A section is devoted to discuss the finite difference formulation. To support the validity of the solution, two examples are included: one demonstrates the exact solution test and the other exhibits an application that had been considered by others.

AN APPLICABLE YALE SPARSE MATRIX TECHNIQUE

A linear system of the form

$$Ax = f \tag{1}$$

can be solved by two classes of methods, the direct method and the iterative (indirect) method, where A is a square, nonsingular matrix of order N, and x and f are vectors. All direct methods employ the Gaussian elimination procedure, which is very suitable for dense systems but has limited usefulness for solving sparse systems because excessive memory storage is required for large N. This is where the sparse matrix technique plays an important and useful role in obtaining an efficient solution. These large sparse matrices usually come from the Method of Lines (MOL) discretization of partial differential equations. There are many techniques introduced to solve the sparse system and an overview of recent developments of these methods can be found in Ref. 5 (Elman). Among these methods, a particular effective method applicable to solve our three-dimensional wide angle underwater acoustic wave equation is the Yale University sparse matrix technique package.[5] One of the sparse techniques contained in the package is known as the Conjugate Gradient (CG) method,[5,6] which has been developed to solve symmetric, positive-definite systems iteratively with great efficiency. In theory, these iterative methods must converge and must converge fast for efficiency. In practice, conventional iterative methods require the estimate of some kind of parameter (e.g., the extreme eigenvalues of the matrix operator A) for fast convergence. Without this estimate one has no idea how fast his applicable

iterative technique converges. This is a drawback of most iterative methods. The CG method minimizes a certain norm in each step and is in a sense optimal over a class of iterative methods. Since the system is sparse, the operations are inexpensive and easy to implement. All these properties make the CG a strong candidate as one of the most robust, rapid convergent iterative methods. This is the reason we introduce it to solve the wide angle three-dimensional partial differential equation. In application, the CG method is effective for solving symmetric, postive-definite problems. In fact, the partial differential equation governing the ocean wave propagation with wide angle capability does not always result in a positive system. On the contrary, it results in a complex system. The CG method cannot be used unless an effective preconditioning technique is applied. The efficiency of the application of the CG method to solve the 3D wide angle PE can be enhanced by preconditioning. These preconditioning techniques solve the system

$$Ax = f$$

by an equivalent system

$$Q^{-1}Ax = Q^{-1}f \quad , \tag{2}$$

where Q^{-1} is in a sense an approximation of A^{-1} so that Eq. (2) can be solved very economically because the actual operation of $Q^{-1}A$ need not be performed explicitly, and at the same time the condition number of A is improved. Since our resulting MOL discretization of the 3D wide angle PE is neither a real system, nor has the positive-definiteness property, we use the

A^* for Q^{-1} as the preconditioning matrix. We then extend the technique to handle a complex, nonsymmetric system whose solution is to be shown effective. The method we consider here is recognized as the application of the CG method to the normal equation.

We begin by dealing with the solution of the system of equations of the form of Eq. (1), i.e.,

$$Ax = f \quad ,$$

where A is a nonsingular, square matrix with complex elements. This problem is equivalent to the normal equation,

$$A^*Ax = A^*f \quad , \tag{3}$$

where A^* is the complex conjugate of A. This suggests that one natural way to solve a nonsymmetric system is by applying preconditioning to the original system and solving the equivalent system (3), provided no extra work is introduced.

In theory, when the CG method is applied to solve system (3), the iterate x_i minimizes the residual norm.[5] One member of the CG family that can be used to solve system (3) is known as the Craig's method[7] and was proposed by Hestenes.[8] In this implementation, the iterate x_i minimizes the residual norm. This is the method we used for our underwater applications and we further extend this application to complex arithmetic.

CRAIG'S ALGORITHM

The computation of the Craig's method involves 5 steps, i.e.,

$$a_i = (r_i,r_i)/(p_i,p_i) \quad ,$$

$$x_{i+1} = x_i + a_i p_i \quad ,$$

$$r_{i+1} = r_i - a_i A p_i \quad ,$$

$$b_i = (r_{i+1},r_i)/(r_i,r_i) \quad ,$$

$$p_{i+1} = A^* r_{i+1} + b_i p_i \quad ,$$

where $r_0 = f - Ax_0$, x_0 is chosen arbitrarily, and $p_0 = A^* r_0$. The above loop is repeated starting with i = 0 until convergence. The work per loop requires 5N multiplications, plus 2 matrix-vector products. Besides, only 4N storages are required for the vectors x, r, p, and Ap.

When dealing with the solution of system (1), we apply the preconditioning technique to transform system (1) into system (3). Then, Craig's method is used to solve system (3). It is natural to think about the need for explicit computation of A*A. The advantage of using Craig's method is that the A*A need not be carried out explicitly. This has been clearly demonstrated in the computation procedure.

THE 3-DIMENSIONAL WIDE ANGLE WAVE EQUATION

As we mentioned in the previous section, there exist two different types of three-dimensional wave equations as a result of the PE approximation. One

is the three-dimensional parabolic wave equation (the 3D PE), originally used by Tappert[1] to derive the standard two-dimensional PE. Solution to the 3D PE has been developed by Baer and Perkins[2] using the Split-step algorithm. The second type is the three-dimensional wide angle partial differential equation, developed by Siegmann, Lee, and Kriegsmann.[3] (We refer to this equation as the 3D wide angle PE.) We chose to concentrate on the solution to the 3D wide angle PE because the 3D PE is a special case. We want to remark why we are motivated to solve the 3D wide angle PE instead of 3D PE; in particular the application of the Yale sparse technique. In this event, the vertical angle of propagation is roughly larger than 15°, due to the irregular nonzero boundary conditions, or other environmental properties where the fast Fourier transform (FFT) is not easily applicable, this is why a general purpose solution is needed.

Now, consider the 3D wide angle PE.

$$\frac{\partial}{\partial r} u = \left(-ik_0 + ik_0 \frac{1 + p_1 \left[n^2(r,\theta,z) - 1 + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} \right] + p_2 \frac{1}{(k_0 r)^2} \frac{\partial^2}{\partial \theta^2}}{1 + q_1 \left[n^2(r,\theta,z) - 1 + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} \right] + q_2 \frac{1}{(k_0 r)^2} \frac{\partial^2}{\partial \theta^2}} \right) u \quad , (4)$$

where $n(r,\theta,z)$ is the index of refraction and k_0 is the reference wavenumber.

Note that when $p_1 = p_2 = 1/2$ and $q_1 = q_2 = 0$, Eq. (4) reduces exactly to the 3D PE.[2] Using the split-step algorithm to solve Eq. (4) is not easily applicable. One can easily see that an alternate general purpose technique is needed to solve Eq. (4). One approach that was considered was to

multiply both sides of Eq. (4) by the operator in the denominator of the right-hand side of Eq. (4). The following was obtained

$$\left\{1 + q_1\left[n^2(r,\theta,z) - 1 + \frac{1}{k_o^2}\frac{\partial^2}{\partial z^2}\right] + q_2\frac{1}{(k_or)^2}\frac{\partial^2}{\partial\theta^2}\right\}\frac{\partial}{\partial r}\,u$$

$$= ik_o\left\{(p_1-q_1)\left[n^2(r,\theta,z) - 1 + \frac{1}{k_o^2}\frac{\partial^2}{\partial z^2}\right] + (p_2-q_2)\frac{1}{(k_or)^2}\frac{\partial^2}{\partial\theta^2}\right\}u \quad , \tag{5}$$

Eq. (5) is not a PE, but a third order partial differential equation known as the pseudo-differential equation. (A reminder to the reader here is that Eq. (4) is called the 3D wide angle PE because the 3D PE is a special case and the terminology PE is a very familiar term.)

In solving Eq. (5), St. Mary and Lee[9] attempted to seek a finite difference solution. Their analyses indicate a restrictive stability condition. For this reason, we attempted a similar implicit finite difference scheme as used for the 3D wide angle PE because of its favorable unconditional stability. The solution by means of an iterative technique is the main topic of this paper; moreover, the efficient solution by means of the Yale sparse technique will be the main result.

DIFFERENCE EQUATION FORMULATION OF THE 3D WIDE ANGLE PE

We are concerned with the solution of the 3D wide angle PE, Eq. (4). We seek such solution by means of the Yale sparse technique, in particular, Craig's method. To deal with the solution of Eq. (4), we must first discuss

the solution procedure as to how to bring Eq. (4) into the finite difference equation such that it is in an easy and acceptable form for Craig's method. Before this formulation, we have a few definitions to state.

Let m indicate the index in the z direction; $\Delta z = h$ indicates the z-increment. Similarly, is used to indicate the index in the θ-direction; $\Delta\theta = \delta$ is the θ-increment; k is used to indicate the range step Δr; and n is used to indicate the range level. Also, for brevity, define

$$x = n^2(r,\theta,z) - 1 + \frac{1}{k_o^2}\frac{\partial^2}{\partial z^2} \quad , \tag{6}$$

and

$$y = \frac{1}{(k_o r)^2}\frac{\partial^2}{\partial\theta^2} \quad . \tag{7}$$

Then, Eq. (4) can be expressed in a short expression using the above definitions, i.e.,

$$\frac{\partial}{\partial r} u = \left(-ik_o + ik_o \frac{1 + p_1 x + p_2 y}{1 + q_1 x + q_2 y}\right) u \quad . \tag{8}$$

Write

$$\mathcal{L} = \left(-ik_o + ik_o \frac{1 + p_1 x + p_2 y}{1 + q_1 x + q_2 y}\right) , \tag{9}$$

then, Eq. (8) can be written in a short operator form, i.e.,

$$\frac{\partial}{\partial r} u = \mathcal{L} u \quad . \tag{10}$$

Numerical solution to Eq. (10) can be expressed as:

$$u^{n+1} = e^{k\frac{\partial}{\partial r}} u^n \quad . \tag{11}$$

Using a half-half splitting of the exponential, and setting up the solution to Eq. (11) by the Crank-Nicolson scheme (an implicit finite difference scheme), we find that an implicit finite difference discretization to Eq. (4) becomes

$$\left[1 - \frac{1}{2}\, k\mathcal{K}\right] u^{n+1} = \left[1 + \frac{1}{2}\, k\mathcal{K}\right] u^{n} \quad . \tag{12}$$

Using the definition of $\mathcal{K}$, x, and y, Eq. (12) becomes

$$\begin{aligned}
&\left[1 + q_1\left(n^2(r,\theta,z) - 1 + \frac{1}{k_o^2}\frac{\partial^2}{\partial z^2}\right) + q_2\,\frac{1}{k_o^2\,(r+k)^2}\frac{\partial^2}{\partial \theta^2}\right] u^{n+1} \\
&- \frac{1}{2}\, ik_o\, k\left\{(p_1-q_1)\left(n^2(r,\theta,z) - 1 + \frac{1}{k_o^2}\frac{\partial^2}{\partial z^2}\right) + (p_2-q_2)\,\frac{1}{k_o^2\,(r+k)^2}\frac{\partial}{\partial \theta^2}\right\} u^{n+1} \\
&= \left[1 + q_1\left(n^2(r,\theta,z) - 1 + \frac{1}{k_o^2}\frac{\partial^2}{\partial z^2}\right) + q_2\,\frac{1}{k_o^2\,r^2}\frac{\partial^2}{\partial \theta^2}\right] u^{n} \\
&+ \frac{1}{2}\, ik_o\, k\left\{(p_1-q_1)\left(n^2(r,\theta,z) - 1 + \frac{1}{k_o^2}\frac{\partial^2}{\partial z^2}\right) + (p_2-q_2)\,\frac{1}{k_o^2\,r^2}\frac{\partial^2}{\partial \theta^2}\right\} u^{n} \quad .
\end{aligned} \tag{13}$$

Using central differences for both operators $\frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial \theta^2}$ in Eq. (13), and simplifying, we obtain

$$\begin{aligned}
&\left(1 + q_1(n^2-1) - \frac{2q_1}{k_o^2}\frac{1}{h^2} - \frac{2q_2}{k_o^2}\frac{1}{(r+k)^2}\frac{1}{\delta^2} - \frac{i}{2}\, k_o\, k(p_1-q_1)(n^2-1)\right. \\
&\left. + i\frac{k}{k_o}\frac{1}{h^2}(p_1-q_1) + i\frac{k}{k_o}\frac{1}{(r+k)^2}\frac{1}{\delta^2}(p_2-q_2)\right) u^{n+1}_{m,\ell}
\end{aligned}$$

$$+\left(\frac{q_1}{k_o^2}\frac{1}{h^2} - \frac{i}{2}\frac{k}{k_o}(p_1-q_1)\frac{1}{h^2}\right)u^{n+1}_{m+1,\ell} + \left(\frac{q_1}{k_o^2}\frac{1}{h^2} - \frac{i}{2}\frac{k}{k_o}(p_1-q_1)\frac{1}{h^2}\right)u^{n+1}_{m-1,\ell}$$

$$+\left(\frac{q_2}{k_o^2}\frac{1}{(r+k)^2}\frac{1}{\delta^2} - \frac{i}{2}\frac{k}{k_o}(p_2-q_2)\frac{1}{(r+k)^2}\frac{1}{\delta^2}\right)u^{n+1}_{m,\ell+1}$$

$$+\left(\frac{q_2}{k_o^2}\frac{1}{(r+k)^2}\frac{1}{\delta^2} - \frac{i}{2}\frac{k}{k_o}(p_2-q_2)\frac{1}{(r+k)^2}\frac{1}{\delta^2}\right)u^{n+1}_{m,\ell-1}$$

$$=\left(1 + q_1(n^2-1) - \frac{2q_1}{k_o^2}\frac{1}{h^2} - \frac{2q_2}{k_o^2}\frac{1}{r^2}\frac{1}{\delta^2} + \frac{i}{2}k_o\,k(p_1-q_1)(n^2-1)\right.$$

$$\left. - i\frac{k}{k_o}\frac{1}{h^2}(p_1-q_1) - i\frac{k}{k_o}\frac{1}{r^2}\frac{1}{\delta^2}(p_2-q_2)\right)u^n_{m,\ell}$$

$$+\left(\frac{q_1}{k_o^2}\frac{1}{h^2} + \frac{i}{2}\frac{k}{k_o}(p_1-q_1)\frac{1}{h^2}\right)u^n_{m+1,\ell} + \left(\frac{q_1}{k_o^2}\frac{1}{h^2} + \frac{i}{2}\frac{k}{k_o}(p_1-q_1)\frac{1}{h^2}\right)u^n_{m-1,\ell}$$

$$+\left(\frac{q_2}{k_o^2}\frac{1}{r^2}\frac{1}{\delta^2} + \frac{i}{2}\frac{k}{k_o}(p_2-q_2)\frac{1}{r^2}\frac{1}{\delta^2}\right)u^n_{m,\ell+1}$$

$$+\left(\frac{q_2}{k_o^2}\frac{1}{r^2}\frac{1}{\delta^2} + \frac{i}{2}\frac{k}{k_o}(p_2-q_2)\frac{1}{r^2}\frac{1}{\delta^2}\right)u^n_{m,\ell-1} \quad . \tag{14}$$

This is the large, sparse system we want to solve efficiently.

Let's use some abbreviated symbols to simplify the coefficient in (14).

Define

$$P_{m,\ell} = \left(1 + q_1(n^2-1) - \frac{2q_1}{k_o^2}\frac{1}{h^2} - \frac{2q_2}{k_o^2}\frac{1}{(r+k)^2}\frac{1}{\delta^2}\right) - i\left[\frac{1}{2}k_o\,k(p_1-q_1)(n^2-1)\right.$$

$$\left. - \frac{k}{k_o}\frac{1}{h^2}(p_1-q_1) - \frac{k}{k_o}\frac{1}{(r+k)^2}\frac{1}{\delta^2}(p_2-q_2)\right] \quad ,$$

$$Q = \frac{q_1}{k_o^2}\frac{1}{h^2} - i\,\frac{1}{2}\frac{k}{k_o}(p_1-q_1)\frac{1}{h^2} \ ,$$

$$R = \frac{q_2}{k_o^2}\frac{1}{(r+k)^2}\ \frac{1}{\delta^2} - i\,\frac{1}{2}\frac{k}{k_o}(p_2-q_2)\frac{1}{(r+k)^2}\ \frac{1}{\delta^2}$$

$$P^+_{m,\ell} = \left(1 + q_1(n^2-1) - \frac{2q_1}{k_o^2}\frac{1}{h^2} - \frac{2q_2}{k_o^2}\frac{1}{r^2}\frac{1}{\delta^2}\right) + i\left[\frac{1}{2}k_o\ k(p_1-q_1)(n^2-1)\right.$$
$$\left. - \frac{k}{k_o}\frac{1}{h^2}(p_1-q_1) - \frac{k}{k_o}\frac{1}{r^2}\frac{1}{\delta^2}(p_2-q_2)\right] \ ,$$

$$R^+ = \frac{q_2}{k_o^2}\frac{1}{r^2}\frac{1}{\delta^2} + i\,\frac{1}{2}\frac{k}{k_o}(p_2-q_2)\frac{1}{r^2}\frac{1}{\delta^2} \ . \qquad (15)$$

We can see Eq. (15) in a simpler form, i.e.,

$$P_{m,\ell}\ u^{n+1}_{m,\ell} + Qu^{n+1}_{m+1,\ell} + Qu^{n+1}_{m-1,\ell} + Ru^{n+1}_{m,\ell+1} + Ru^{n+1}_{m,\ell-1}$$
$$= P^+_{m,\ell}\ u^n_{m,\ell} + Q^*\ u^n_{m+1,\ell} + Q^*\ u^n_{m-1,\ell} + R^+\ u^n_{m,\ell+1} + R^+\ u^n_{m,\ell-1} \ , \qquad (16)$$

where Q^* means the complex conjugate of Q. $P_{m,\ell}$ and $P^+_{m,\ell}$ matrices depend upon the variation in r,θ, and z. Q matrix is constant in all 3 variables. R and R^+ matrices are dependent on the range variable only.

AN ILLUSTRATION

For illustrative purposes, we use a simple example to display Eq. (16) in a matrix form. In general, $m = 1, 2, \ldots, M$ and $= 1, 2, \ldots, L$. Note that $m = 0$ indicates the surface boundary; $m = M + 1$ indicates the bottom boundary; and $\ell = 0$ and $\ell = L + 1$ will be explained below.

We start with assigning $m = 1, 2, 3, 4$ and $\ell = 1, 2, 3$ at the initial range level n and march to the next range $n + 1$. In this example, $L = 3$ and $\ell = 1, 2, 3$ mean there are three sectors, as shown in Figure 1. In computation, we must deal with the indexes $\ell = 0$ and $\ell = 4$. Since the index is periodic with a period $L = 3$, then, ℓ can be regarded as $\ell = \ell \pmod 3$. Therefore, $\ell = 0$ is the same as $\ell = 3$, and $\ell = 4$ is the same as $\ell = 1$, as also shown in Figure 1.

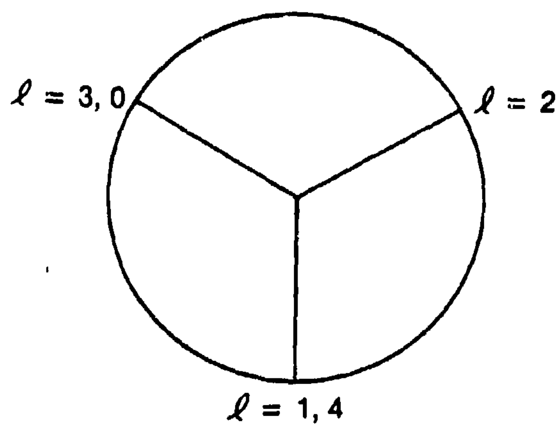

Figure 1. Azimuthal Sectors

Now we make an attempt to put Eq. (16) in a matrix form encountering the boundary conditions. We use the convention $\ell = \ell \pmod 3$ to express ℓ_0 and ℓ_4 by ℓ_3 and ℓ_1, respectively. We then can construct a matrix in a general form making use of the periodic boundary condition. We find

$$
\begin{bmatrix}
P_{1,1} & Q & 0 & 0 & R & 0 & 0 & 0 & R & 0 & 0 & 0 \\
Q & P_{2,1} & Q & 0 & 0 & R & 0 & 0 & 0 & R & 0 & 0 \\
0 & Q & P_{3,1} & Q & 0 & 0 & R & 0 & 0 & 0 & R & 0 \\
0 & 0 & Q & P_{4,1} & 0 & 0 & 0 & R & 0 & 0 & 0 & R \\
R & 0 & 0 & 0 & P_{1,2} & Q & 0 & 0 & R & 0 & 0 & 0 \\
0 & R & 0 & 0 & Q & P_{2,2} & Q & 0 & 0 & R & 0 & 0 \\
0 & 0 & R & 0 & 0 & Q & P_{3,2} & Q & 0 & 0 & R & 0 \\
0 & 0 & 0 & R & 0 & 0 & Q & P_{4,2} & 0 & 0 & 0 & R \\
R & 0 & 0 & 0 & R & 0 & 0 & 0 & P_{1,3} & Q & 0 & 0 \\
0 & R & 0 & 0 & 0 & R & 0 & 0 & Q & P_{2,3} & Q & 0 \\
0 & 0 & R & 0 & 0 & 0 & R & 0 & 0 & Q & P_{3,3} & Q \\
0 & 0 & 0 & R & 0 & 0 & 0 & R & 0 & 0 & Q & P_{4,3}
\end{bmatrix}
\begin{bmatrix}
u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \\ u_{4,3}
\end{bmatrix}^{n+1}
+
\begin{bmatrix}
Qu_{0,1} \\ 0 \\ 0 \\ Qu_{5,1} \\ Qu_{0,2} \\ 0 \\ 0 \\ Qu_{5,2} \\ Qu_{0,3} \\ 0 \\ 0 \\ Qu_{5,3}
\end{bmatrix}^{n+1}
=
$$

$$
\begin{bmatrix}
P^*_{1,1} & Q^* & 0 & 0 & R^* & 0 & 0 & 0 & R^* & 0 & 0 & 0 \\
Q^* & P^*_{2,1} & Q^* & 0 & 0 & R^* & 0 & 0 & 0 & R^* & 0 & 0 \\
0 & Q^* & P^*_{3,1} & Q^* & 0 & 0 & R^* & 0 & 0 & 0 & R^* & 0 \\
0 & 0 & Q^* & P^*_{4,1} & 0 & 0 & 0 & R^* & 0 & 0 & 0 & R^* \\
R^* & 0 & 0 & 0 & P^*_{1,2} & Q^* & 0 & 0 & R^* & 0 & 0 & 0 \\
0 & R^* & 0 & 0 & Q^* & P^*_{2,2} & Q^* & 0 & 0 & R^* & 0 & 0 \\
0 & 0 & R^* & 0 & 0 & Q^* & P^*_{3,2} & Q^* & 0 & 0 & R^* & 0 \\
0 & 0 & 0 & R^* & 0 & 0 & Q^* & P^*_{4,2} & 0 & 0 & 0 & R^* \\
R^* & 0 & 0 & 0 & R^* & 0 & 0 & 0 & P^*_{1,3} & Q^* & 0 & 0 \\
0 & R^* & 0 & 0 & 0 & R^* & 0 & 0 & Q^* & P^*_{2,3} & Q^* & 0 \\
0 & 0 & R^* & 0 & 0 & 0 & R^* & 0 & 0 & Q^* & P^*_{3,3} & Q^* \\
0 & 0 & 0 & R^* & 0 & 0 & 0 & R^* & 0 & 0 & Q^* & P^*_{4,3}
\end{bmatrix}
\begin{bmatrix}
u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \\ u_{4,3}
\end{bmatrix}^{n}
+
\begin{bmatrix}
Q^*u_{0,1} \\ 0 \\ 0 \\ Q^*u_{5,1} \\ Q^*u_{0,2} \\ 0 \\ 0 \\ Q^*u_{5,2} \\ Q^*u_{0,3} \\ 0 \\ 0 \\ Q^*u_{5,3}
\end{bmatrix}^{n}
$$

In general, the large sparse system to be solved is in the form

$$A\, u^{n+1} = B\, u^{n} + u_0^{n+1} + u_0^{n} \quad , \tag{18}$$

where u_0^{n+1} contains surface and bottom boundary information at the advanced range level and u_0^n contains surface and bottom boundary

information at the present range level. The A and B matrixes possess the format

$$
\begin{bmatrix}
T & D & 0 & \cdots & 0 & 0 & R \\
D & T & D & \cdots & 0 & 0 & 0 \\
0 & D & T & \cdots & 0 & 0 & 0 \\
 & & & \vdots & & & \\
0 & 0 & 0 & \cdots & T & D & 0 \\
0 & 0 & 0 & \cdots & D & T & D \\
R & 0 & 0 & \cdots & 0 & D & T
\end{bmatrix} \tag{19}
$$

All the block matrices (T, D, and R) are of the same order MxM. Each T matrix is tridiagonal; whereas each off-diagonal block matrix D and R are diagonal matrices. Entire matrices are a 7-diagonal matrix with the property that $A = A^T$ and $B = B^T$.

The right-hand side of Eq. (18) can be carried out by one matrix-vector operation and two vector additions. Eq. (18) is a large, sparse system, which we want to solve by taking advantage of the Yale sparse technique.

Note that if we consider that the wave propagates all around a complete 360°, we deal with a system where A and B are of the form (19), i.e., a 7-diagonal matrix. If we consider that the wave propagates only in a sector, then the periodic boundary condition for the azimuthal plane disappears, and we then solve system (18) where A and B are in a simpler form as below:

$$\begin{bmatrix} T & D & 0 & \dots & 0 & 0 & 0 \\ D & T & D & \dots & 0 & 0 & 0 \\ 0 & D & T & \dots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & T & D & 0 \\ 0 & 0 & 0 & \dots & D & T & D \\ 0 & 0 & 0 & \dots & 0 & D & T \end{bmatrix} \tag{20}$$

which is a 5-diagonal matrix.

Further, if we consider that the wave propagates only in a vertical plane, this reduces to a two-dimensional case. We then deal with the system (18) where A and B are tridiagonal matrices, i.e.,

$$\begin{bmatrix} T & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & T & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & T & \dots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & T & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & T & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & T \end{bmatrix} \tag{21}$$

It is important to note that when $p_1 = p_2 = /2$ and $q_1 = q_2 = 0$, the system (16) reduces to the 3D PE.

NUMERICAL RESULTS

As a test of accuracy, the Yale sparse technique (Craig's method) was programmed on VAX 11/780 computer to solve Eq. (9) using the system of equations expressed by Eq. (16). We used a known exact solution below as an accuracy check.

To describe the test procedure, we express Eq. (9) in the form of Eq. (5), i.e.,

$$\left\{1 + q_1\left[\left(n^2(r,\theta,z) - 1\right) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right] + q_2\frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial\theta^2}\right\}\frac{\partial}{\partial r}\,u$$

$$= ik_0\left\{(p_1-q_1)\left[\left(n^2(r,\theta,z) - 1\right) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right] + (p_2-q_2)\frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial\theta^2}\right\}u \quad .$$

We look for a solution to Eq. (5) in the form

$$u(r,\theta,z) = \sin(\Omega z)\, e^{im\theta}\,\phi(r) \quad . \tag{22}$$

Substituting Eq. (22) into Eq. (5), we find

$$\left\{1 + q_1\left[\left(n^2(r,\theta,z) - 1\right) - \frac{\Omega^2}{k_0^2}\right] - \frac{q_2 m^2}{k_0^2}\,\frac{1}{r^2}\right\}\phi_r$$

$$= ik_0\left\{(p_1-q_1)\,\left[\left(n^2(r,\theta,z) - 1 - \frac{\Omega^2}{k_0^2}\right] - \frac{(p_2-q_2)m^2}{k_0^2}\,\frac{1}{r^2}\right\}u \quad . \tag{23}$$

We select $n^2(r,\theta,z) - 1 - \frac{\Omega^2}{k_o^2} = 0$ for computational simplicity. Since $k = k_o n(r,\theta,z) = \omega/c$, then

$$k_o = \left[\left(\frac{\omega}{c}\right)^2 - \Omega^2\right]^{1/2} . \tag{24}$$

Eq. (23) can be simplified using the k_o defined by Eq. (24) to give

$$\frac{\partial\phi}{\partial r} = \left(\frac{-ik_o(p_2-q_2)\ m^2/(k_o r)^2}{1 - q_2 m^2/(k_o r)^2}\right)\phi = -if(r)\phi \quad , \tag{25}$$

which is a first order ordinary differential equation.

The solution to Eq. (25) can readily be expressed in the form

$$\phi = A\ e^{i\int f(r)\ dr} . \tag{26}$$

The effort needed to find the $\phi(r)$ is the evaluation of the $\int f(r)\ dr$. We use

$$u(r_o,\theta,z) = \sin(\Omega z)e^{im\theta}\ \phi(r_o) \tag{27}$$

as the initial field;

$$u(r,\theta,z_o) = \sin(\Omega z_o)e^{im\theta}\ \phi(r) = 0 \tag{28}$$

for the surface condition boundary; and

$$u(r,\theta,z_B) = \sin(\Omega z_B)e^{im\theta}\ \phi(r) = 0 \tag{29}$$

for the bottom boundary condition. These boundary conditions are particularly selected such that

$$\Omega z_0 = 0 \text{ and } \Omega z_B = \text{an integer multiple of } \pi.$$

The initial range is selected to start at 50 m so that the farfield approximation is valid.

The azimuthal plane is divided into 10 sectors at 36° each. Since there are 10 sectors and we partition the depth into 199 increments, then we solve a system of equations of the size 1990 x 1990 dealing with a 7-diagonal matrix. The results presented below are a display of boundaries between two adjacent sectors. Not only do we compare the actual computed numerical complex numbers with the exact solution but the dB values as well.

Case 1: Small Angle propagation ($p_1 = p_2 = 1/2$, $q_1 = q_2 = 0$)

An evaluation of the $\int f(r)\, dr$ gives $-m^2/(2k_0 r)$. This produces the solution

$$\phi(r) = A\, e^{i\frac{m^2}{2k_0 r}} \quad . \tag{31}$$

Table 1 describes the results; the first row indicates the computed values and the second row indicates the exact solution. The results are taken at the boundary between the third and the fourth sectors at 108° at a range of 50.4 m.

Table 1. Results of Small Angle Propagation

I	Z(I)	LOSS	u(I)	
3	30.00	12.636	(0.18834E+00	-0.13793E+00)
3	30.00	12.636	(0.18886E+00	-0.13722E+00)
6	60.00	6.859	(0.36627E+00	-0.26824E+00)
6	60.00	6.859	(0.36729E+00	-0.26685E+00)
9	90.00	3.749	(0.52397E+00	-0.38372E+00)
9	90.00	3.749	(0.52541E+00	-0.38174E+00)
12	120.00	1.841	(0.65270E+00	-0.47800E+00)
12	120.00	1.841	(0.65451E+00	-0.47553E+00)
15	150.00	0.688	(0.74537E+00	-0.54587E+00)
15	150.00	0.688	(0.74743E+00	-0.54304E+00)
18	180.00	0.108	(0.79685E+00	-0.58357E+00)
18	180.00	0.108	(0.79906E+00	-0.58055E+00)

Case 2: Wide Angle Propagation ($p_1 = p_2 = 3/4$, $q_1 = q_2 = 1/4$)

An evaluation of the $\int f(r)\,dr$ gives the solution

$$\phi(r) = A\, e^{-i \frac{m(p_2-q_2)}{2\sqrt{q_2}} \ln \frac{k_o r - m\sqrt{q_2}}{k_o r + m\sqrt{q_2}}} \quad . \qquad (32)$$

Numerical results are presented in the same manner as in Case 1. These results are given in Table 2.

Table 2. Results of Wide Angle Propagation

I	Z(I)	LOSS	u(I)	
3 3	30.00 30.00	12.645 12.636	(0.22578E+00 (0.22613E+00	-0.58389E+00) -0.57994E+00)
6 6	60.00 60.00	6.868 6.859	(0.43904E+00 (0.43976E+00	-0.11362E+00) -0.11268E+00)
9 9	90.00 90.00	3.757 3.749	(0.62819E+00 (0.62909E+00	-0.16245E+00) -0.16134E+00)
12 12	120.00 120.00	1.852 1.841	(0.78225E+00 (0.78365E+00	-0.20236E+00) -0.20098E+00)
15 15	150.00 150.00	0.694 0.688	(0.89377E+00 (0.89492E+00	-0.23150E+00) -0.22952E+00)
18 18	180.00 180.00	0.117 0.108	(0.95544E+00 (0.95673E+00	-0.24602E+00) -0.24537E+00)

From the solution results, we examine the behavior of the solution of Case 2 for large $k_o r$. First, we consider the real part of the solution for

$$x = \frac{m(p_2-q_2)}{2\sqrt{q_2}} \ \ln\left[\frac{k_o r - m\sqrt{q_2}}{k_o r + m\sqrt{q_2}}\right] ,$$

$$\cos(x) = \cos\left\{\frac{m(p_2-q_2)}{2\sqrt{q_2}}\left[\left[-\frac{m\sqrt{q_2}}{k_o r} - \frac{1}{2}\frac{m^2 q_2}{(k_o r)^2} - \cdots\right\}\right.\right.$$

$$\left.\left.-\left\{\frac{m\sqrt{q_2}}{k_o r} - \frac{1}{2}\frac{m^2 q_2}{(k_o r)^2} + \cdots\right]\right]\right\} .$$

Then,

$$\cos(x) \approx \cos\left\{\frac{m(p_2-q_2)}{2\sqrt{q_2}}\left[-\frac{2\sqrt{q_2}m}{k_o r}\right]\right\} = \cos\left\{-\frac{m^2(p_2-q_2)}{k_o r}\right\} \tag{33}$$

for large k_0r. For wide angle parameters $p_2 - q_2 = 1/2$, Eq. (33) reduces to the real part of the solution of Case 1.

In examining the $\cos\left(\frac{m^2}{2\,k_0 r}\right)$, we note that the function increases monotonically after $k_0 = m^2/\pi$. It approaches unity as $k_0 r \to \infty$. When the function is close to unity, three-dimensional effects are lost, and Eq. (4) behaves like the two-dimensional parabolic wave equation below

$$\frac{\partial}{\partial r} u = \left(-ik_0 + ik_0 \frac{1 + p\left[n^2(r,z) - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right]}{1 + q\left[n^2(r,z) - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right]}\right)u \quad .$$

The more rapid the azimuthal variation (i.e., the bigger m^2), the further out in range the three-dimensional effects influence the solution.

An application is also presented here as a second example. This example has been solved by Perkins and Baer,[11] using the Split-step algorithm for three dimensions. The sound speed profile is taken from a Pacific profile such that $c(r,\theta,z) = c_m(z) + (0.001)r\sin\theta$, where $c_m(z)$ takes on the tabulated values below.

Table 3. A Pacific Profile

z(m)	c(z) (m/s)
0.00	1536.500
152.400	1539.243
406.300	1501.143
1015.9	1471.882
5587.91	1549.606
5587.91	1555.526

This profile has a large linear gradient in the cross range direction; the gradient is 1 m/s per km. The profile in the vertical plane at 0° is a typical profile in the North Pacific Ocean.

The source is placed at 254 m below the surface with a source frequency of 25 Hz. We calculated the propagation loss up to a maximum range of 140 km. We choose to present below the results on one particular sector at 0°. Along with the 3D wide angle PE solution plot is the graphical result of Perkins and Baer[11] for comparison. The propagation loss reading at 120 km for the same receiver depth is approximately 90 dB, showing satisfactory agreement with the known result. The 3D wide angle PE result is presented in figure 2; and the Perkins-Baer result is presented in figure 3.

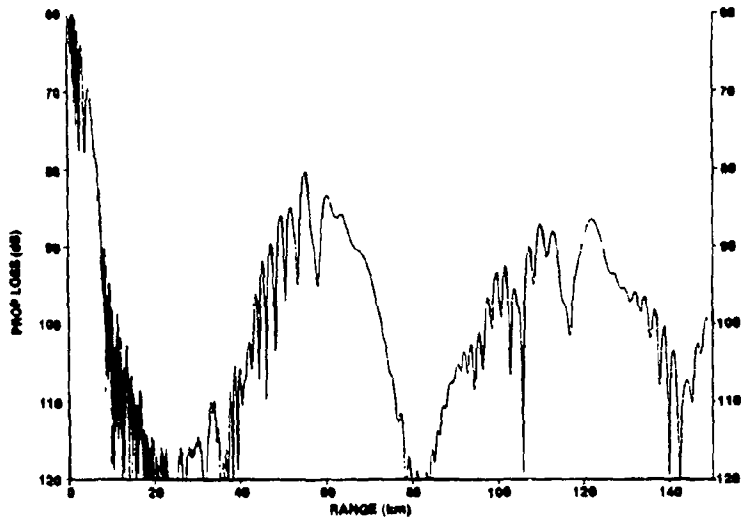

Figure 2. Propagation Loss vs. Range at Zero Degree Azimuthal Angle

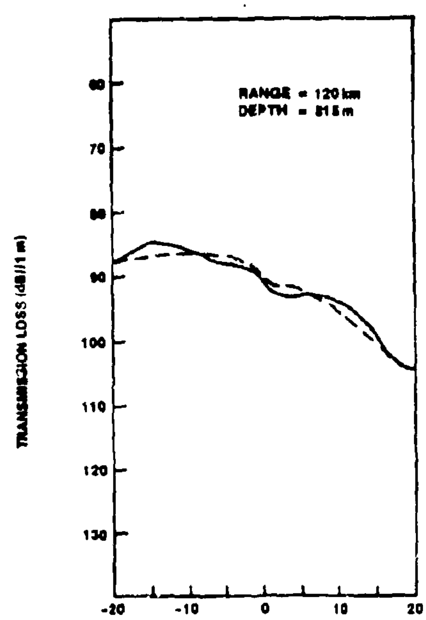

Figure 3. Propagation Loss as a Function of Azimuthal Angle Between -20° and 20° at a Range 120 km and a Depth of 315 m.

CONCLUSIONS

In situations where FFT is applicable, the FFT can do well. Under the same situation, the CG method can also solve the same problem with the same accuracy, but the computation speed is not competitive with the FFT computation. The solution to the wide angle three-dimensional partial differential equation cannot be directly solved by the FFT; this is a definite advantage of the Yale sparse technique. The applicable CGNE that we used here requires 5N multiplications per loop plus two matrix-vector products, and only 4N storage locations are required for the vector operations.

Since CG is an iterative technique dealing with inner products, it is desirable to implement the procedure in a vectorized machine. This is another advantage of the Yale sparse technique.

The numerical solutions produced in this paper demonstrated the general purpose capability of the Yale sparse technique. Even though the solution is accurate, the present solution can by no means be regarded as the most efficient solution. It is believed that a clever preconditioning technique can be developed to enhance the efficiency of our applications.

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5. A RANGE REFRACTION PARABOLIC EQUATION

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ABSTRACT: Application of the standard parabolic wave equation to solve real problems requires a clever selection of the reference wavenumber k_0. An extended parabolic equation (PE) having range refraction capability is reintroduced to be totally independent of k_0. The existing implicit finite difference (IFD) model was applied to test the range refraction PE. Results compare favorably with known solutions for weakly range-dependent environments, but yield significant corrections for propagation through strong oceanic fronts.

INTRODUCTION

The RAnge REfraction Parabolic Equation (RAREPE), introduced by Tappert [1] over a decade ago, has important acoustic effects but has been neglected. Since the standard parabolic equation (PE) is performing satisfactorily, users have not paid attention to the RAREPE. Besides, there did not exist efficient algorithms directly applicable to solve the RAREPE. Now, we have the implicit finite-difference (IFD [2]) package, and the effort required to modify the available IFD code to solve the RAREPE is inexpensive. To give a complete understanding of the RAREPE, we first summarize the derivation of the RAREPE, then describe how we solved the RAREPE by the finite difference solution. A special section is devoted to discuss a set of illustrative examples. These examples are used to show (1) the close agreement between the standard PE and the RAREPE if there is no front, (2) the important property, independence of k_0, of the RAREPE; and (3) the range refraction effects by weak, moderate, and strong fronts.

DERIVATION OF THE RANGE REFRACTION PE

We start with the two-dimensional reduced wave equation, i.e.,

$$p_{rr} + \frac{1}{r} p_r + p_{zz} + k_0^2 n^2(r,z)p = 0 \quad . \tag{1}$$

Setting $p(r,z) = \frac{\hat{u}(r,z)}{\sqrt{r}}$ and applying the farfield approximation, $k_0 r \gg 1$, we find Eq. (1) becomes

$$\hat{u}_{rr} + \hat{u}_{zz} + k_0^2 n^2(r,z)\hat{u} = 0 \quad . \tag{2}$$

Write Eq. (2) in the form

$$\left(\frac{\partial^2}{\partial r^2} + k_0^2 Q^2\right)\hat{u} = 0 \quad , \tag{3}$$

where

$$Q^2 = n^2(r,z) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} \quad . \tag{4}$$

We assume that the dependence of $n(r,z)$ on the range variable r is weak such that $\frac{\partial n(r,z)}{\partial r}$ is negligible, but we shall later pick up the neglected terms. This assumption allows the operator $\frac{\partial}{\partial r}$ to commute with Q^2. We can now factor Eq. (3) into two equations:

$$i\,\frac{\partial \hat{u}_+}{\partial r} + k_0\, Q\hat{u}_+ = 0 \quad , \tag{5}$$

and

$$-i\,\frac{\partial \hat{u}_-}{\partial r} + k_0\, Q\hat{u}_- = 0 \quad . \tag{6}$$

Then, the solution field $\hat{u}(r,z)$ is just the combination of the outgoing wave $\hat{u}_+(r,z)$ and incoming wave $\hat{u}_-(r,z)$.

The operator Q, defined by Eq. (4), is actually

$$Q = \left(1 + (n^2(r,z) - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)^{1/2} \quad . \tag{7}$$

In this paper, we deal with the small angle PE, i.e., we approximate the Q by

$$Q \cong 1 + \frac{1}{2}\left\{(n^2(r,z) - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right\} \quad , \tag{8}$$

and make use of the usual envelope definition $u = \hat{u}(r,z)\, e^{ik_0 r}$, which leads to

the standard PE originally introduced by Tappert [1], i.e.,

$$\frac{\partial u}{\partial r} = \frac{i}{2} k_0 (n^2(r,z) - 1)\, u + \frac{i}{2k_0} \frac{\partial^2 u}{\partial z^2} \quad . \tag{9}$$

To make the local error small, we need

$$||n^2(r,z) - 1|| \ll 1 \quad ,$$

and

$$\left|\left|\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2}\right|\right| \ll 1 \quad .$$

A detailed discussion of the estimate of $||n^2(r,z) - 1||$ and $||\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2}||$ can be found in reference 1.

To keep track of the relative errors made in the course of calculation, we can monitor the size of $||n^2(r,z) - 1||$ and $||\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2}||$ and keep them both small. We now present a modified PE, which requires that only one of the $\left\{n^2(r,z) - 1\right\}$ and $\left\{\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2}\right\}$ be small, as occurring in Q, but is of order unity. This modified PE is capable of dealing with a large range variation of the index of refraction.

Consider two operators A and B, where $A = n^2(r,z)$ and $B = \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2}$. In general A and B do not commute. We expand $(A + \delta B)^{1/2}$ by the formula

$$(A + \delta B)^{1/2} = A^{1/2} + \delta C + o(\delta^2) \quad , \tag{10}$$

where

$$C = \int_0^\infty e^{-\sqrt{A}s}\, B\, e^{-\sqrt{A}s}\, ds \quad . \tag{11}$$

The proof of (10) is given in reference 1.

We now apply Eqs. (10) and (11) to the operator Q given by (7).

For small $\frac{1}{k_o^2}\frac{\partial^2}{\partial z^2}$, we obtain

$$Q = 1 + \sqrt{(n^2(r,z) - 1)} + \int_0^\infty e^{-n(r,z)s}\left(\frac{1}{k_o^2}\frac{\partial^2}{\partial z^2}\right)e^{-n(r,z)s}\, ds \quad . \tag{12}$$

It can be found that

$$\begin{aligned} Qu(r,z) &= n(r,z)\, u + \frac{1}{k_o^2}\int_0^\infty e^{-2ns}\left(u_{zz} - sn_{zz}\, u - 2sn_z u_z + s^2 n_z^2 s\right) ds \\ &= n(r,z)\, u + \frac{1}{2k_o^2}\left[\left(\frac{u_z}{n}\right)_z + \frac{1}{2}\left(\frac{n_z^2}{n^3} - \frac{n_{zz}}{n^2}\right) u\right] \quad . \end{aligned} \tag{13}$$

Substituting Eq. (13) into Eq. (5) for the outgoing wave, we obtain

$$i\,\frac{\partial u}{\partial r} + \frac{1}{2k_o}\frac{\partial}{\partial z}\left(\frac{1}{n(r,z)}\frac{\partial u}{\partial z}\right) + k_o\left[n(r,z) + \frac{1}{4k_o^2}\left(\frac{n_z^2}{n^3} - \frac{n_{zz}}{n^2}\right)\right]u = 0 \quad . \tag{14}$$

Eq. (14) is the RAREPE and is valid to all orders in $(n^2(r,z) - 1)$. Equation (14) is approximately equal to the standard PE, Eq. (9), when n(r,z)

is range-independent and is close to unity; in this case the $\frac{n^2(r,z) - 1}{2}$ in the standard PE can be replaced by $(n(r,z) - 1)$.

It ought to be noted that Eq. (14) and Eq. (9) are the same if the $n(r,z)$ is a constant in both r and z variables, and $n(r,z) = 1$. Substituting the constant $n(r,z)$ into Eq. (14), we find

$$\frac{\partial u}{\partial r} = ik_o\, n(r,z)\, u + \frac{i}{2k_o}\frac{\partial^2 u}{\partial z^2} \quad . \tag{15}$$

Eq. (15) can be put in the form

$$\frac{\partial \psi}{\partial r} = ik_o(n^2(r,z) - 1)\psi + \frac{1}{2k_o}\frac{\partial^2 \psi}{\partial z^2} \quad , \tag{16}$$

and replacing $(n(r,z) - 1)$ by $(1/2)(n^2(r,z) - 1)$, we find

$$\frac{\partial \psi}{\partial r} = \frac{i}{2}\, k_o(n^2(r,z) - 1)\psi + \frac{i}{2k_o}\frac{\partial^2 \psi}{\partial z^2} \quad . \tag{17}$$

We see that Eq. (17) is exactly in the same format as Eq. (9).

We now have established the relationship between (15) and (16) based on the transformation

$$u(r,z) = \psi(r,z)\, e^{ik_o r} \quad . \tag{18}$$

We now proceed to show that solutions to Eq. (15) and Eq. (16) are identical in magnitude in order to establish the k_o independence.

Substituting Eq. (18) into Eq. (15), we find

$$\psi_r = ik_o(n(r,z) - 1)\psi + \frac{i}{2k_o}\psi_{zz} \quad , \tag{19}$$

which is identical to Eq. (16). From the relationship of Eq. (18), it is easily seen that

$$||u||^2 = ||\psi||^2 \quad .$$

This shows that the solution of (15) is equivalent to the solution of (18) in magnitude independent of k_o.

THE REFERENCE WAVENUMBER k_o

We return now to the standard, small angle PE, which takes the form

$$i\frac{\partial u}{\partial r} + \frac{1}{2k_o}\frac{\partial^2 u}{\partial z^2} + \frac{k_o}{2}(n^2(r,z) - 1)u = 0 \quad . \tag{20}$$

It is clear that Eq. (20) is k_o-dependent, thus, different k_o's lead to different solutions. Obviously, there is only one k_o associated with a given set of environmental conditions that will produce the solution closest to the real solution. We have been confronting the problem of how to select the best k_o. In fact, PE users never have to worry about the k_o-selection because existing PE models, such as the split-step code [3] and the IFD package [2], all offer the option to have a default k_o value if the user is not certain what k_o to use. The default k_o is chosen from k_o =

$2\pi f/c_0$, where c_0 is the reference speed and is selected to be the average sound speed in the water column. The user can choose k_0 to be the sound speed at the source level for range-independent environments. Moreover, for range-dependent environments and for range-dependent sound speed profiles, the user can ask that an interpolation of the sound speed profile be performed within each range interval and apply the same procedure to select c_0 as in the range-independent case. These choices so far present no big problem; and make the selection of the k_0 ignorable. Pierce [4] re-emphasized the importance of the k_0 selection and introduced a formula to determine the range-dependent k_0 based on the Rayleigh quotient. Some numerical experiments have been carried out at NUSC, New London Laboratory. The results show some phase effects on k_0 variation. A detailed study of the k_0-selection is going to be reported separately when it is completed. All these facts strongly suggest the desirability of having either a variable k_0 PE selection or a k_0-independent PE. This paper chooses to deal with the latter.

THE FINITE DIFFERENCE FORMULATION AND SOLUTION

Rewrite Eq. (14) in the form

$$i \frac{\partial u}{\partial r} + \frac{1}{2k_0} \frac{\partial}{\partial z} \left(\frac{1}{n} \frac{\partial}{\partial z} \right) + k_0 \, vu = 0 \quad , \tag{21}$$

where

$$v = n + \frac{1}{4k_0} \left(\frac{n_z^2}{n^3} - \frac{n_{zz}}{n^2} \right) \quad . \tag{22}$$

$$v_m = n_m\left[1 + \frac{1}{4k_o^2\; n_m^4\; h^2}\left(n_m^2 - n_{m+1}\; n_{m+1}\right)\right] \quad . \tag{27}$$

Using tne definition $k_o n = \frac{\omega}{c_o}\,\frac{c_o}{c} = \frac{\omega}{c} = k$, (27) can be written as

$$v_m = \frac{c_o}{c_m}\left[1 + \frac{c_m^4}{4\omega^2\; h^2}\left(\frac{1}{c_m^2} - \frac{1}{c_{m+1}\; c_{m-1}}\right)\right] \quad . \tag{28}$$

Substituting (24) through (28) into Eq. (21), we find

$$\frac{\partial u_m}{\partial r} = \alpha_m\; u_{m+1} + \beta_m\; u_m + \gamma_m\; u_{m-1} \quad , \tag{29}$$

where

$$\alpha_m = \frac{i}{\omega h^2\left(\frac{1}{c_{m+1}} + \frac{1}{c_m}\right)} \quad ,$$

$$\gamma_m = \frac{i}{\omega h^2\left(\frac{1}{c_m} + \frac{1}{c_{m-1}}\right)} \quad ,$$

and

$$\beta_m = -(\alpha_m + \gamma_m) + i\,\frac{\omega}{c_m}\left[1 + \frac{c_m^2}{4\omega^2\; h^2}\left(1 - \frac{c_m^2}{c_{m+1}\; c_{m-1}}\right)\right] \quad .$$

Note that $\alpha_m^* = -\alpha_m$ and $\gamma_m^* = -\gamma_{m-1}$. We see that

$$\frac{\partial}{\partial r}|u_m|^2 = \alpha_m\left(u_m^*\; u_{m+1} - u_{m+1}\; u_m\right) + \gamma_m\left(u_m^*\; u_{m-1} - u_{m-1}^*\; u_m\right) \quad .$$

This implies that

$$\frac{\partial}{\partial r}\left(\sum_m |u_m|^2\right) = \sum_m \alpha_m\left(u_m^*\; u_{m+1} - u_{m-1}^*\; u_m\right) + \sum_m \gamma_m\left(u_m^*\; u_{m-1} - u_{m-1}^*\; u_m\right)$$

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$$= \sum_m \left(\alpha_m - \gamma_{m+1}\right)\left(u_m^* \, u_{m+1} - u_{m+1}^* \, u_m\right) .$$

But, $\alpha_m = \gamma_{m+1}$. Therefore

$$\frac{\partial}{\partial r}\left(\sum_m |u_m|^2\right) = 0 \Rightarrow \sum_m |u_m|^2 = \text{constant} .$$

This implies that the finite difference scheme is energy conservative.

Next, from Eq. (29), we set up the Crank-Nicolson difference equation as follows:

$$\begin{aligned}
&- \frac{k}{2} \alpha_m^{n+1} u_{m+1}^{n+1} + \left(1 - \frac{k}{2} \beta_m^{n+1}\right) u_m^{n+1} - \frac{k}{2} \gamma_m^{n+1} u_{m-1}^{n+1} \\
&= \frac{k}{2} \alpha_m^n u_{m+1}^n + \left(1 + \frac{k}{2} \beta_m^n\right) u_m^n + \frac{k}{2} \gamma_m^n u_{m-1}^n .
\end{aligned} \tag{30}$$

This is the exact IFD format recognized by the IFD model. The solution by the IFD code becomes easy.

NUMERICAL ILLUSTRATIONS

For qualitative information, this section presents three examples which are used to show the various effects of range refraction.

We use the following cannonical profile whose input parameters are defined and tabulated below:

$$c(r,z) = c_A[1 + \varepsilon(e^{-\eta} - 1 + \eta)] \quad , \tag{31}$$

where

$$\eta = \frac{z - z_A}{(B/2)} \quad , \tag{32}$$

c_A = sound speed of axis,

z = depth variable,

z_A = depth of axis of sound channel, and

B = thickness of thermal front.

We assign an ocean depth of 5 km and assume the ocean bottom to be flat. We calculate the propagation loss up to 140 km in range. We place the source at 100 m below the surface with a source frequency of 100 Hz.

Define

$$B(r) = B_1 + \frac{B_2 - B_1}{2}\left[1 + \tanh\left(\frac{r - r_F}{L}\right)\right] \quad , \tag{33}$$

where r_F is the range at which the front occurs,

L is the length of the front, and

B_1, B_2 are parameters.

In addition, define

$$\varepsilon(r) \quad \frac{Bg}{2c_A} \quad , \tag{34}$$

where $g = 2c_A / B$. (35)

PROFILE	B_1 (km)	B_2 (km)	Z_A (km)	r_F (km)	L (km)	REMARKS
1	1.2	1.2	1	50, 60	-	No front
2	1.2	1.0	1	50, 60	20	moderate front
3	1.2	0.8	1	50, 60	20	strong front

The cannonical profiles for these different examples are described in figure 1.

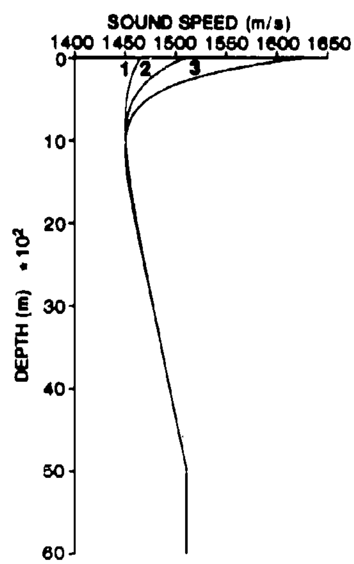

Figure 1. Cannonical Profiles

The examples (using the above three profiles) are executed using the following information:

EXAMPLE	INITIAL RANGE (km)	INITIAL PROFILE	FINAL RANGE (km)	FINAL PROFILE	RANGE FRONT (km)
1	0	1	140	1	No
2	0	1	140	2	60
3	0	1	140	3	60

The following set of graphs (figure 2) were obtained by the IFD model using the input information of Profile 1.

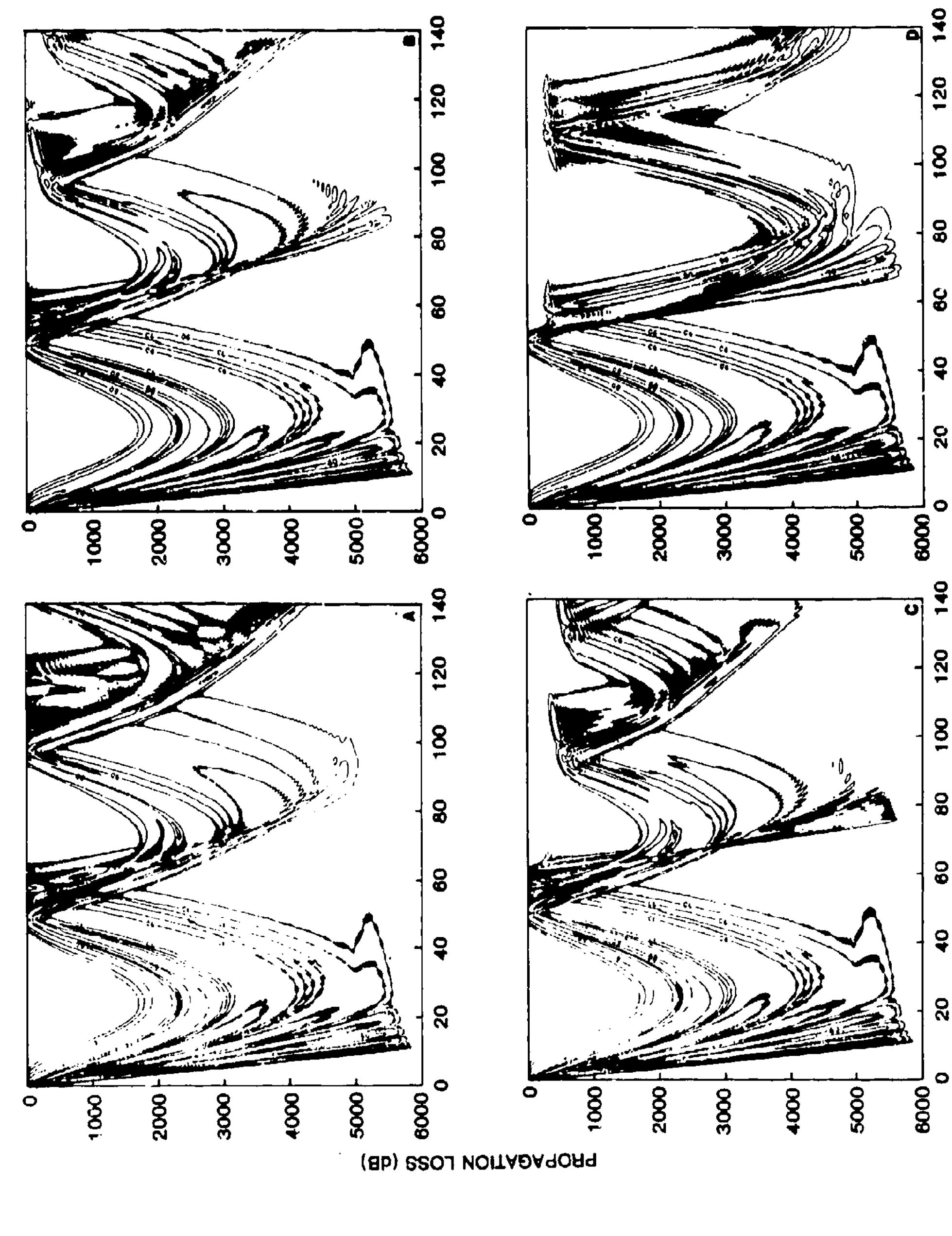

Figure 2. Contour Plots

The results have the following meanings:

CASE	RANGE FRONT AT km RANGE	FRONT	SOLVED BY
A	-	No	Standard PE
B	60	Moderate	RAREPE
C	60	Strong	RAREPE
D	50	Strong	RAREPE

Figure 3 presents the propagation loss curves over the range interval [0, 140 km] for three different receiver depths. There is no front present in these examples. The left column displays the standard PE results; the right column displays the results with range refraction. Notice that the differences among these results are very small. In order to make the difference more visible, we must group the results together in a magnified plot.

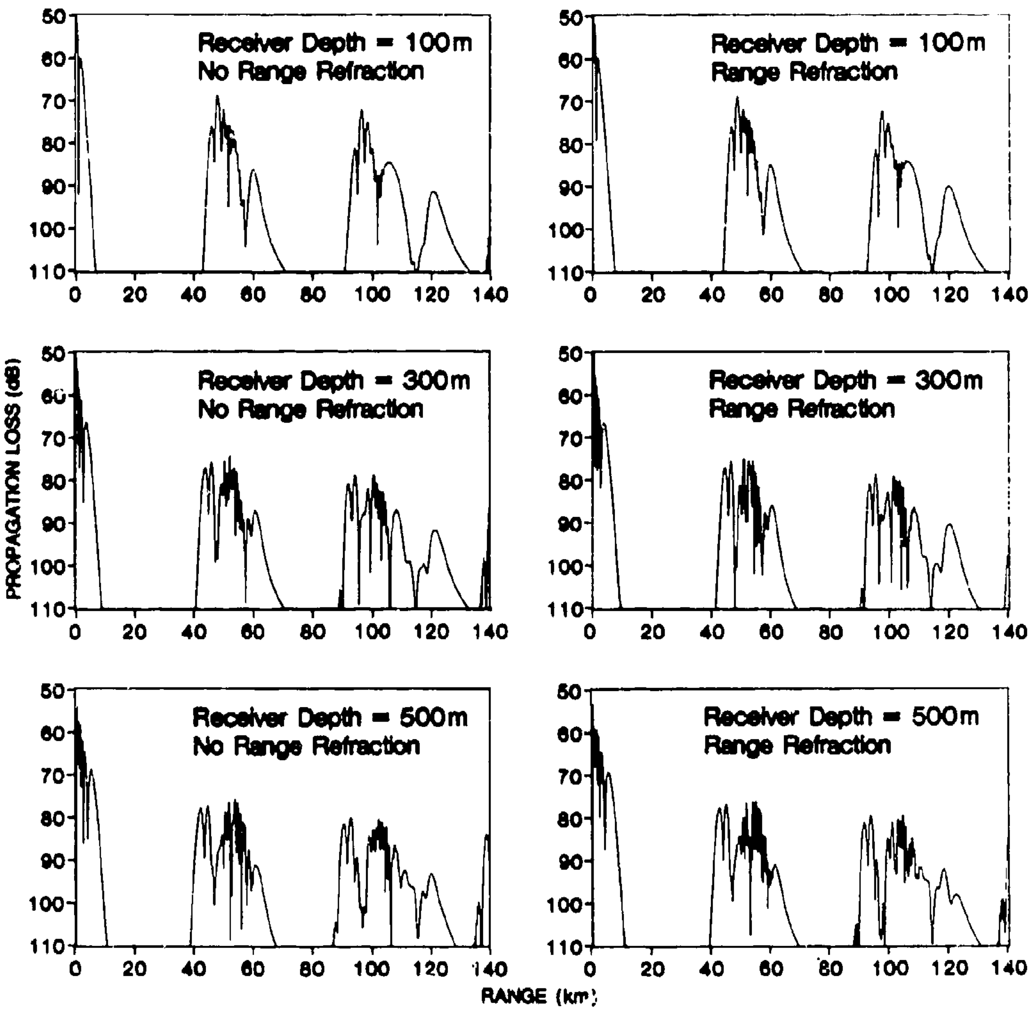

Figure 3. Results Without Range Front

The original Helmholtz equation is k_0-independent. For a range-independent problem, we use an accurate fast field program (FFP [3]) solution as a reference for comparison. The split-step [1] solution is also included. Figure 4 contains the solutions produced by the IFD, the split-step, and the FFP, with and without range refraction effects.

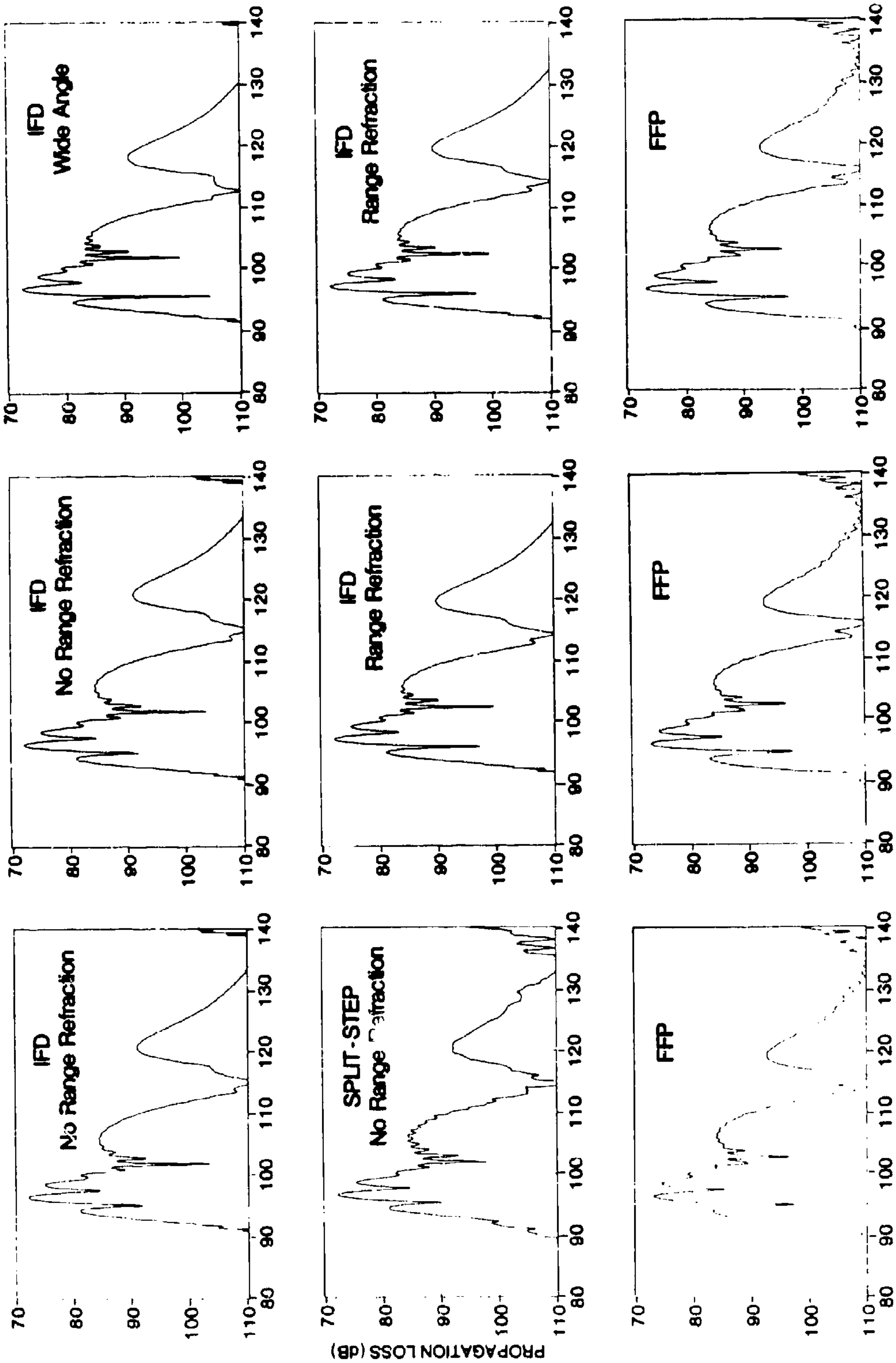

Figure 4. Results Comparison

From the comparison, a little difference is shown among the IFD, split-step, and the FFP solutions. When comparing results among IFD, IFD range refraction, and the FFP, also a little difference is shown. However, the comparisons among the IFD wide angle, IFD range refraction, and the FFP, we experience a difference between the IFD wide angle and the IFD range refraction; this is expected because the present IFD wide angle model accommodates the range refraction. This is going to be pointed out in the next example.

Similar as the set of no range refraction results, the next set (figure 5) consists of propagation loss curves over the range interval [0, 140 km] with a strong range front that occurs at the range of 60 km.

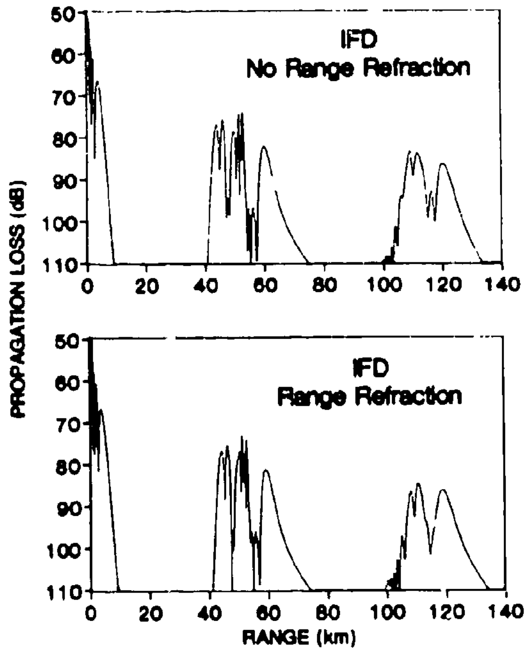

Figure 5. Results With a Strong Range Front

Whenever the range front is present, especially the strong front, the RAREPE results are more meaningful than the standard PE results. Results on figure 6 show a difference between the RAREPE and the wide angle PE. Both the standard PE and the Wide Angle PE were formulated dependent upon the special approximation of the square root operator $\sqrt{1 + \varepsilon + \mu}$, where $\varepsilon = n^2(r,z) - 1$, and $\mu = \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2}$. The IFD solution considers the square root to be approximated in the form $\frac{1 + p(\varepsilon + \mu)}{1 + q(\varepsilon + \mu)}$. When $p = \frac{1}{2}$ and $q = 0$, the resulting PE is the standard PE. When $p = 3/4$ and $q = 1/4$, the resulting PE is the wide angle PE. Expanding $(1 + q(\varepsilon + \mu)^{-1} (1 + p(\varepsilon + \mu)) = (1 - q(\varepsilon + \mu) + q^2(\varepsilon + \mu)^2 + \ldots) (1 + p(\varepsilon + \mu)) = 1 + (p - q)(\varepsilon + \mu) - q(p - q)(\varepsilon + \mu)^2 +$ high order terms. Notice that the portion $(\varepsilon + \mu)^2$ represents the wide angle and $(\varepsilon + \mu)^2 = \varepsilon^2 + \mu^2 + \varepsilon\mu + \mu\varepsilon$, where the last two terms represent the range refraction. Therefore, the present wide angle PE IFD version accommodates the range refraction.

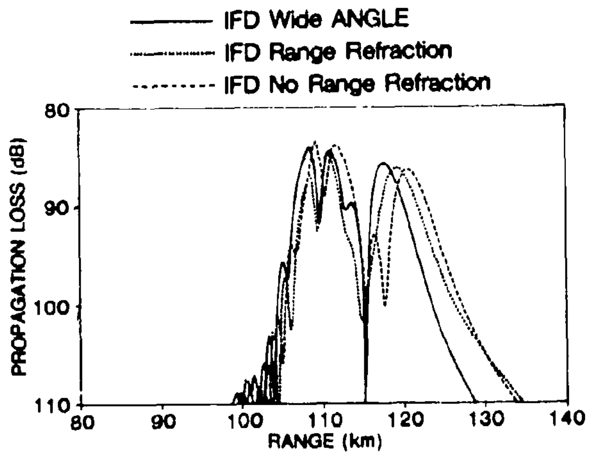

Figure 6. Results Comparison

CONCLUSIONS

A parabolic equation, having range refraction capability, is re-introduced. This range refraction PE is independent of the choice of k_0, and is also more efficient to handle a rapidly varying index of refraction on the range variable. Numerical results show that the range refraction PE is useful for strong range fronts. Under such environments, the range refraction PE has an advantage over the standard PE. However, a price has to be paid on the computation speed.

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6. THE HYBRID PARABOLIC EQUATION -- A RAY MODEL

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ABSTRACT: Using the standard parabolic equation to solve high frequency problems is impractical because of excessive running time. A new HYbrid Parabolic Equation using Ray theory (HYPER) is developed to handle high frequency problems. This paper discusses the theory and development of the HYPER discussed in detail.

INTRODUCTION

Successful applications of the standard parabolic equation (PE) [1] have been evident. However, using the standard PE to solve high frequency problems suffers an excessive running time [2]. To overcome this difficulty, a new HYbrid Parabolic Equation using the Ray theory (HYPER) is designed to be particularly effective in handling high frequency problems. A complete discussion is given to describe how the HYPER is derived. Within the discussion, how the ray equation is obtained will also be described in full.

THEORETICAL BACKGROUND AND DEVELOPMENT

Prior to the development of the high frequency parabolic equation (HYPER), it is necessary to list a set of definitions below.

Let $u(r,z)$ be the pressure field, $n(r,z)$ be the index of refraction, k_o be the reference wavenumber, c_o be the reference sound speed, and $c(r,z)$ be the sound speed profile.

The standard PE takes the general expression

$$i \frac{\partial u}{\partial r} + \frac{1}{2k_o} \frac{\partial^2 u}{\partial z^2} - k_o U(r,z) u = 0 \qquad (1)$$

for small propagation angles, and $U(r,z) = \frac{1}{2}(1-n^2(r,z)) = \frac{1}{2}\left(1 - \frac{c_o^2}{c^2(r,z)}\right)$.

An observation of Eq. (1) stirs up two motivations that lead to the development of the HYPER

Motivation 1: The standard PE, Eq. (1), has no limit on frequency, but the equation itself is strongly frequency dependent. As a function of frequency, the computation time is excessive; because of this we need a more effective way to handle high frequency problems. We feel that there is no reason why we cannot develop a high frequency PE that possesses the same format as (1).

Motivation 2: The geometry of the acoustics is independent of frequency. It is highly desirable to have a ray PE model independent of frequency but with full range effects. We, thus, seek to develop a high frequency PE in format (1) but totally frequency independent.

Based on these motivations, we proceed to develop the HYPER. Motivation 1 suggests that the HYPER takes the expression

$$i \frac{\partial \tilde{\mu}}{\partial r} + \frac{1}{2k_o} \frac{\partial^2 \tilde{\mu}}{\partial \zeta^2} - k_o V(r,\zeta) \tilde{u} = 0 \quad , \tag{2}$$

where $V(r,\zeta)$ is related to $U(r,z)$ by the relationship

$$V(r,\zeta) = U(r,z_R(r) + \zeta) - U(r,z_R(r)) - \zeta \frac{\partial}{\partial z} U(r,z_R(r)) \quad , \tag{3}$$

where

$$u(r,z) = \tilde{u}(r,\zeta)\, e^{ik_o\left[S_o(r) + \zeta \dot{z}_R(r)\right]} \quad , \tag{4}$$

$$z = a_R(r) + \zeta \quad ,$$

the dot "•" indicates the r-derivative; $S_0(r)$ is taken for granted as a function of r for the time being and is defined later in the section on the development of ray equation (the particular ray equation associated with (2)). Now, we proceed to show how (2) was derived. Write

$$u(r,z) = A(r,z)\, e^{ik_0 S(r,z)} \quad . \tag{5}$$

For large k_0, substitute (5) into the standard PE, Eq. (1), and use asymptotic expansions for k_0 (keeping order up to $o(1/k_0)$), we find

$$\frac{\partial S}{\partial r} + \frac{1}{2}\left(\frac{\partial S}{\partial z}\right)^2 + U = o\left(\frac{1}{k_0}\right) \quad . \tag{6}$$

Equation (6) is an inhomogeneous, nonlinear partial differential equation (PDE) called the Hamilton-Jacoby equation. Rays of the PDE (6) are the characteristics of the PDE (6). To solve eq. (6), differentiate (6) with respect to z, we find

$$\frac{\partial^2 S}{\partial r \partial z} + \frac{\partial S}{\partial z}\,\frac{\partial^2 S}{\partial z^2} + \frac{\partial U}{\partial z} = 0 \quad .$$

Then, let $\theta = \partial S/\partial z$ and substitute it into the above equation, we obtain

$$\frac{\partial \theta}{\partial r} + \theta\,\frac{\partial \theta}{\partial z} + \frac{\partial U}{\partial z} = 0 \quad . \tag{7}$$

Write

$$\frac{d\theta}{dr} = \frac{\partial \theta}{\partial r} + \theta\,\frac{\partial \theta}{\partial z} \quad ,$$

but,

$$\frac{d\theta}{dr} = \frac{\partial \theta}{\partial r} + \frac{dz}{dr}\,\frac{\partial \theta}{\partial z} \quad .$$

Comparing the above 2 equations, θ must equal dz/dr. Solving (7) by the method of characteristics, we have on the curve z(r)

$$\frac{dz}{dr} = \theta \quad , \tag{8}$$

and

$$\frac{d\theta}{dr} + \frac{dU}{dz} = 0 \quad . \tag{9}$$

Combining (8) and (9) gives the HYPER ray equation, i.e.,

$$\frac{d^2z}{dr^2} = -\frac{\partial}{\partial z} U(r,z) \quad . \tag{10}$$

Suppose we trace a particular ray (i.e., a particular solution $z = z_R(r)$). To deal with the standard PE, let $\zeta = z - z_R(r)$. Corresponding to the operator $\frac{\partial}{\partial r}$, we find $\frac{\partial}{\partial r} - z_R \frac{\partial}{\partial \zeta}$; similarly $\frac{\partial}{\partial \zeta}$ for $\frac{\partial}{\partial z}$. The standard PE can be transformed into

$$i \frac{\partial u}{\partial r} - i\dot{z}_R \frac{\partial u}{\partial} + \frac{1}{2k_o} \frac{\partial^2 \mu}{\partial \zeta^2} - k_o \; U(r, z_R(r) + \zeta)\, \mu = 0 \quad . \tag{11}$$

Now, recall the $S_o(r)$. We express

$$S_o(r) = S(r, z_R(r)) \quad , \tag{12}$$

and we refresh our memory that the $S(r, z_R(r))$ satisfies the Hamilton-Jacoby equation (6).

From the relationship of θ, we see that

$$\theta = \frac{dS}{dz} = \frac{dz}{dr} \quad .$$

Then,

$$S(r,z) = \int \frac{\partial S}{\partial z}\, dz + \int \frac{\partial S}{\partial r}\, dr = \int \left(\frac{\partial S}{\partial z}\frac{dz}{dr} + \frac{\partial S}{\partial r}\right) dr$$

$$= \int \left[\left(\frac{\partial S}{\partial z}\right)^2 + \frac{\partial S}{\partial r}\right] dr$$

$$= \int \left[\frac{1}{2}\left(\frac{\partial S}{\partial z}\right)^2 - U(r,z)\right] dr$$

$$= \int \left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 - U(r,z)\right] dr \quad ,$$

which implies that

$$S_0(r) = \int_0^r \left[\frac{1}{2}\left(\frac{dz_R}{dr'}\right)^2 - U\left(r', z_R(r')\right)\right] dr' \quad . \tag{13}$$

This is the complete expression for $S_0(r)$. We need the r-derivative of $S_0(r)$. We find that

$$\frac{d}{dr} S_0(r) = \frac{1}{2}\left(\frac{dz_R}{dr}\right)^2 - U(r, z_R(r)) \tag{14}$$

Now, substitute expression (4) into Eq. (2), we obtain the desired HYPER, i.e.,

$$i\,\frac{\partial \tilde{u}}{\partial r} + \frac{1}{2k_0}\frac{\partial^2 \tilde{u}}{\partial \zeta^2} - k_0 \Big[U(r, z_R(r) + \zeta) - U(r, z_R(r))$$

$$- \zeta \frac{\partial}{\partial z} U(r, z_R(r))\Big]\, \tilde{u} = 0 \quad . \tag{15}$$

Next, we shall show how (15) was obtained making use of all the developments. From (4), we find

$$\frac{\partial u}{\partial r} = \left[\frac{\partial \tilde{u}}{\partial r} + ik_o\left(\frac{1}{2}\dot{z}_R^2 - U_R + \zeta\ddot{z}_R\right)\tilde{u}\right] e^{ik_o\left[S_o(r) + \zeta\dot{z}_R(r)\right]} ,$$

$$\frac{\partial u}{\partial \zeta} = \left[\frac{\partial \tilde{u}}{\partial \zeta} + ik_o\,\dot{z}_R\,\tilde{u}\right] e^{ik_o\left[S_o(r) + \zeta z_R(r)\right]} ,$$

and

$$\frac{\partial^2 u}{\partial \zeta^2} = \left[\frac{\partial^2 \tilde{u}}{\partial \zeta^2} + 2\,ik_o\,\dot{z}_R\,\frac{\partial \tilde{u}}{\partial r} - k_o^2(\dot{z}_R)^2\,\tilde{u}\right] e^{ik_o\left[S_o(r) + \zeta\dot{z}_R(r)\right]} .$$

By substituting the above partial derivatives into (2), we obtain (11).

Q.E.D.

COMPUTATIONAL ASPECTS

To perform the computation completely, a number of steps are involved. Each present step depends upon the previous step after the problem is properly started. We discuss the computations required for each step and show the continuity from one step to the next. Most of the present computations are carried in a practical manner to make the solution work. Room exists for future improvement in computations.

Step 1: Calculation of the Ray Equation

One important portion of the computation is the calculation of the ray equation to describe how the ray is traced. The ray trace portion requires the implementation of the ray equation, (10), i.e.,

$$\frac{d^2z}{dr^2} = -\frac{\partial U}{\partial z} \quad .$$

The ray equation is a second order ordinary differential equation (ODE) and can be treated as a purely initial value problem. This second order ODE can be solved efficiently by an existing convergent numerical ODE, equation such as the methods given in reference 3. We apply the Stomer-Cowell formula to perform the ray trace.

We express our ray equation in a short form as

$$z'' = -\frac{\partial U}{\partial z} = f(r,z) \quad . \tag{16}$$

The Cowell corrector formula takes the form

$$z^C_{n+1} - 2z_n + z_{n-1} = \frac{(\Delta r)^2}{12}\left[f_{n+1} + 10f_n + f_{n-1}\right] \quad . \tag{17}$$

To get the predicted value to integrate formula (17), we use the Stormer predictor formula

$$z^P_{n+1} - 2z_n + z_{n-1} = (\Delta r)^2 \, f_n \quad . \tag{18}$$

Since we are solving an initial value problem, the $z(0)$, $z'(0)$ are known. We also know f_n and z_0, which is $z(0)$. We need to know z_1, which we choose to obtain by the Taylor series expansion, i.e.,

$$z_1 = z(0) + (\Delta r)\; z'(0) + \tfrac{1}{2}(\Delta r)^2\; z''(0) + \ldots$$

$$= z(0) + (\Delta r)\; z'(0) + \tfrac{1}{2}(\Delta r)^2\; f(0) + \ldots$$

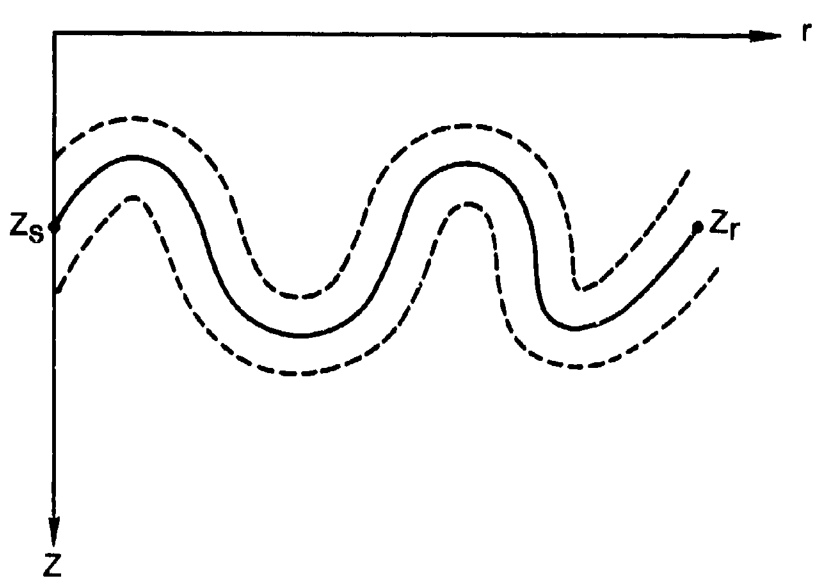
The Taylor expansion has an order of accuracy in truncation error less accurate than the numerical ODE methods used. We overcome this difficulty by using a very small step size.

According to Henrici [3], the selection of the step size to satisfy the corrector's convergence is such that $\Delta r < \left| \frac{1}{\frac{1}{12} L} \right|^{1/2}$, where L is the Lipschitz constant of f(r,z).

To carry out the complete procedure will give $z_R(r)$.

Step 2: Determination of the Ray Trace Region

Next, we make use of the information obtained from the previous step, and attempt to define a strip (called the wide W) covering the ray path. The figure below shows the picture in the r-z plane.

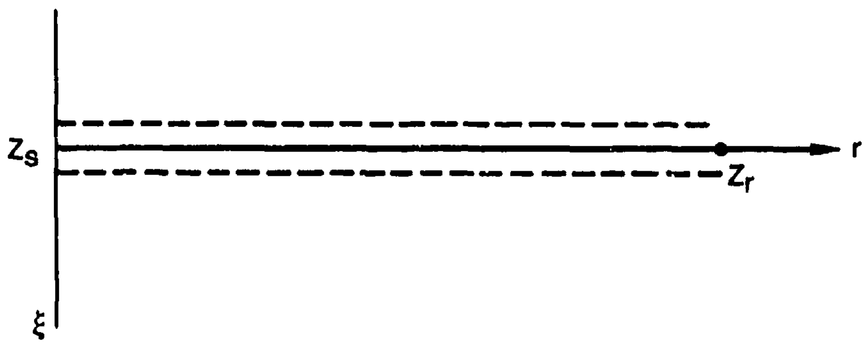


here z_s is the source depth and, z_r is the receiver depth. The region of interest is determined to be

$$z_s - W \leq z \leq z_s + W \quad ,$$

therefore, the width is 2W. We see that if we solve the problem in the original r-z plane, we need to solve the problem in the entire original region; therefore, the amount of work is by no means able to be reduced.

The advantage is evident if we solve the problem in the equivalent r-plane as shown by the figure below.



The region covered in the r-ζ plane is small; therefore, much less work is needed to complete the computation and should be more accurate. This above illustration shows one single ray; for a family of rays, we handle the family by taking the union.

Step 3: Computation of $V(r,\zeta)$

Function $V(r,\zeta)$ is calculated using Formula (3), i.e.,

$$V(r,\zeta) = U(r,z_R(r) + \zeta) - U(r,z_R(r))$$
$$-\zeta \frac{\partial}{\partial z} U(r,z_R(r)) \quad ,$$

where $U(r,z)$ is defined to be

$$U(r,z) = \frac{1}{2}\left(1 - \frac{c_0^2}{c^2(r,z)}\right) \quad .$$

On the ray path, $U(r,z)$ is calculated by

$$U(r,z_R(r)) = \frac{1}{2}\left(1 - \frac{c_0^2}{c^2(r,z_R(r))}\right), \tag{19}$$

and $\frac{\partial}{\partial z} U$ is calculated by

$$\frac{\partial}{\partial z} U(r,z_R(r)) = \frac{1}{\Delta z}\left[U(r,z_R(r) + \Delta z) - U(r,z_R(r))\right] \quad . \tag{20}$$

Step 4: Computation of $S_0(r)$

$S_0(r) = S(r,z_R(r))$ is calculated by formula (13).

Step 5: To Obtain the Solution $u(r,z)$

Our main objective is to obtain the solution of Eq. (1), i.e., $u(r,z)$. We summarize the procedures involved in order to obtain the $u(r,z)$ and show how the $u(r,z)$ is obtained.

- Obtain the ray equation solution $z_R(r)$.
- Using $z_R(r)$, compute $V(r,\zeta)$ by means of $V(r,\zeta)$ - $U(r,z)$ relationship.
- Group a family of rays together that have strong effects between the source and receiver.
- Determine the grid points in the r-ζ plane and set up the numerical solution for $\tilde{u}(r,\zeta)$.
- Solve $\tilde{u}(r,\zeta)$ by the IFD model.
- Solve $S_0(r) = S(r,z_R(r))$ by formula (13).
- Finally,

$$u(r,z) = \tilde{u}(r,\zeta)\, e^{ik_0[S_0(r) + z_R(r)]}$$

CONCLUSIONS

We have developed the HYPER specially for handling high frequency problems. Through an efficient implementation, the advantages of the HYPER are clear--not only is it accurate but also it reduces the execution time tremendously. Thus, the HYPER should be considered to be a practical high frequency model.

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7. A VARIABLE DENSITY PARABOLIC EQUATION

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ABSTRACT: In this paper, we derive a new parabolic equation (PE) that incorporates the effects of a variable ocean density. This density can be smooth or piecewise-smooth. Thus, our model reduces to the standard PE when the density is constant and it alleviates the need for interfacial conditions when the density is stratified in a piecewise fashion. We also present a numerical scheme that will be used to solve our new equation. This difference scheme has a conservation law that is the discrete analog of the new PE's conservation law.

DERIVATION

The propagation of sound in an ocean with variable density ρ is governed by the elliptic equation

$$\rho\nabla \cdot \frac{1}{\rho} \ p + k'^2 n^2 p = 0 \quad , \tag{1}$$

where p is the acoustic pressure, $k' = \omega/c_0$, ω is the frequency of the time harmonic source, c_0 is a reference sound speed, $n = c_0/c$, and c is the sound speed in the ocean. A time dependence of $e^{-i\omega t}$ is suppressed. Equation (1) is to be solved in a spatial domain D', which contains the water. A simple model is obtained by assuming both the ocean bottom and the water-air interface are flat. Specifically,

$$D' = \left\{(x',y',z') \middle| |x'| < \infty, \ |y'| < \infty, \quad 0 \le z' \le H'\right\} \quad ,$$

where the primed variables denote dimensional quantities. Since equation (1) is elliptic, boundary conditions are required to complete the mathematical description of the problem. The conditions used in this report are

$$\frac{\partial p}{\partial z'} = 0, \ z' = H' \tag{2}$$

and

$$p = 0, \ z' = 0 \quad . \tag{3}$$

Thus, the ocean has a hard bottom and a pressure release (free) surface.

The source deriving equation (1) is usually modeled as a point disturbance located at $x' = y' = 0$, $z' = z_0'$. It is omitted from equation (1) for simplicity.

In many underwater applications the domain (in polar coordinates), $D' = \left\{(r',z',\theta) \,\middle|\, 0 \leq r' \leq R',\ 0 \leq z' H',\ 0 \leq \theta \leq 2\pi\right\}$, where equation (1) must be solved is extremely slender. By this we mean the parameter

$$\epsilon = (H'/R')^2 \tag{4}$$

satisfies $\epsilon \ll 1$ where R' is the maximum range of interest. We now introduce the dimensionless variables r and z used by Tappert [1], i.e.,

$$r = \epsilon k'r' \tag{5}$$

and

$$z = \sqrt{\epsilon}\, k'z' \quad . \tag{6}$$

Accordingly D' is transformed into

$$D = \left\{(r,z,\theta) \,\middle|\, 0 \leq r \leq \ell,\ \ 0 \leq z\ \underline{0}\ \ell,\ \ 0 \leq \theta < 2\pi\right\} \quad , \tag{7}$$

where $\ell = (k'H')H'/R'$. We asume that this number is fixed and is order one with respect to the parameter ϵ. Introducing this change of variables into equations (1) – (3) we find the acoustic pressure satisfies

$$\epsilon^2 \left[p_{rr} + \frac{1}{r}\, pr - \frac{1}{\rho}\, \rho_r\, p_r\right] + \epsilon\left[p_{zz} - \frac{\rho_z}{\rho}\, p_z\right] + n^2 p = 0 \quad , \tag{8}$$

$$p = 0, \quad z = 0 \quad , \tag{9}$$

and

$$\frac{\partial p}{\partial z} = 0, \quad z = \ell \quad . \tag{10}$$

In addition to the boundary conditions (9) and (10), we demand that p is bounded as $r \to \ell$.

We now make the assumption that n^2 deviates slightly from a constant and takes the functional form

$$n^2(x',y',z') = 1 + \epsilon f(r,z) \quad . \tag{11}$$

The constant 1 in this equation is arrived at by taking c_0 to be the average of c throughout D. The factor ϵ in (11) demonstrates the weak dependence of c on depth and range. (This apparent minor perturbation creates profound effects on acoustic propagation when the range is as short as a few wavelengths!)

We also assume that the density ρ depends upon the variables r and z in a smooth or piecewise smooth fashion.

When (11) is inserted into (8) we observe the presence of the small parameter ϵ in front of nearly every term. To cavalierly set these terms to zero would render a physically meaningless result. Guided by previous experience with such matters, we apply the method of multiple scales to this equation. Specifically, we assume that

$$p(x',y',z') = P(\xi,r,z;\ \epsilon) \quad , \tag{12}$$

where the fast variable ξ is defined by

$$\xi = r/\epsilon \quad . \tag{13}$$

Inserting this variable and (11) into (8), we obtain the equation

$$\left[P_{\xi\xi} + P\right] + \epsilon\left[2P_{r\xi} + \frac{1}{r}P - \frac{\rho_r}{\rho}P_\xi + P_{zz} - \frac{\rho_z}{\rho}P_z + fP\right]$$
$$+ \epsilon^2\left[P_{rr} + \frac{1}{r}P_r - \frac{\rho_r}{\rho}P_r\right] \quad . \tag{14}$$

The subscripts denote partial differentiation. Next we make the assumption that P has the asymptotic expansion

$$P \sim \sum_{n=0}^{\infty} \epsilon^n P_n(\xi,r,z,\theta) \quad , \quad \epsilon \to 0 \quad . \tag{15}$$

When this expression is inserted into (16) we equate to zero the coefficients of the powers of ϵ. This yields an infinite sequence of equations of which the first two are

$$L\,P_0 \equiv P_{0\xi\xi} + P_0 = 0 \quad , \tag{16}$$

and

$$L\,P_1 = 2P_{0r\xi} + \frac{1}{r}P_{0\xi} - \frac{\rho_r}{\rho}P_{0\xi} + P_{0zz} - \frac{\rho_z}{\rho}P_{0z} + fP_0 \quad , \tag{17}$$

for $0 < z < \ell$, $0 < r < \ell$. Inserting (15) into the boundary conditions (9) and (10) and equating to zero the coefficients of the powers of ε, we obtain an infinite sequence of boundary conditions. The companions for (16) and (17) are

$$P_n = 0,\ z = 0,\ n = 0,1 \quad , \tag{18}$$

and

$$\frac{\partial P_n}{\partial z} = 0,\ z = \ell;\ n = 0,1 \quad . \tag{19}$$

The solution of equation (16) is

$$P_0 = A_0(r,z)\, e^{i\xi} + B_0(r,z)\, e^{-i\xi} \quad , \tag{20}$$

where the amplitudes A_0 and B_0 are functions of the listed variables. Because of the assumed time dependence, $e^{-i\omega t}$, we set

$$B_0(r,z) = 0 \tag{21}$$

as a failure to do so would yield incoming waves from infinity. Inserting (20) and (21) into (17) gives

$$L\, P_1 = \left[2i\, A_{0r} + \frac{i}{r} A_0 - \frac{i\rho_r}{\rho} A_0 + A_{0zz} - \frac{\rho_z}{\rho} A_{0z} + f\, A_0\right] e^{i\xi} \quad , \tag{22}$$

which has the general solution

$$P_1 = A_1(r,z)\, e^{i\xi} + \frac{i}{2} M(A_0)\, e^{i\xi} \quad , \tag{23}$$

where $M(A_0)$ is the bracketed term on the right-hand side of equation (22). We observe that P_1 remains bounded as $\xi = r/\varepsilon \to \infty$ only if

$$M(A_0) = 2i\, A_{0r} + \frac{i}{r} A_0 - \frac{i\rho_r}{\rho} A_0 + A_{0zz} - \frac{\rho_z}{\rho} A_{0z} + f\, A_0 = 0 \; . \tag{24}$$

If we now set

$$A_0(r,z) = \sqrt{\rho(r,z)} \quad \frac{u_0}{\sqrt{r}} \tag{25}$$

into (24), we find that u_0 must satisfy our new variable density parabolic differential equation (VDPE)

$$-\,2i\,\frac{\partial u_0}{\partial r} = \sqrt{\rho}\,\frac{\partial}{\partial z}\left[\frac{1}{\rho}\,\frac{\partial}{\partial z}\left(\sqrt{\rho}\;u_0\right)\right] + f\, u_0 \quad . \tag{26}$$

We now make a few interesting observations about this new equation. First, when ρ is a constant, (26) reduces to the standard parabolic equation [1]. Secondly, the differential operator involving z is symmetric or formally self-adjoint [2]. Thirdly, the quantity

$$E \equiv \int_0^{\ell} |u_0|^2\, dz \tag{27}$$

is independent of range, i.e., $dE/dr \equiv 0$. This follows directly from (26) and the boundary conditions for u_0 are

$$u_0 = 0, \quad z = 0 \quad , \tag{28}$$

and

$$\frac{\partial}{\partial z}\left(\sqrt{\rho}\; u_0\right) = 0, \quad z = \ell \quad . \tag{29}$$

Equations (28) and (29) are direct consequences of (18), (19), and (25). Fourthly, we observe that equation (26) itself was derived without using the boundary conditions given in (18) and (19). Thus, our new parabolic equation will hold even when more realistic boundary conditions are implemented. Finally, equation (26) can be used even when ρ is piecewise smooth. This will allow us to study interfaces that are not planar or straight lines. In this sense our new parabolic equation extends the analysis given by Lee and McDaniels [3,4].

A CONSERVATIVE FINITE DIFFERENCE SCHEME

In this section, we present a finite difference scheme, which is second order accurate in depth and first order accurate in range, for solving equation (26). This difference scheme will conserve a discrete analog of $dE/dr = 0$, where E is given by (27). The method of analysis and other examples are given by Kriegsmann and Mahar [5].

We begin by rewriting (26) as

$$-2i\,\frac{\partial u_0}{\partial r} = a\,\frac{\partial}{\partial z}\left[b\,\frac{\partial}{\partial z}\,(au_0)\right] + fu_0 \quad , \tag{30}$$

where $a \equiv \sqrt{\rho}$ and $b \equiv 1/\rho$. Setting $u_j^n \equiv u_0(r_n, z_j)$, it is easily verified by Taylor's theorem that

$$a \frac{\partial}{\partial z}\left[b\frac{\partial}{\partial z}(au_0)\right]_{(r_n, z_j)} = \hat{L}(u_j^n)(\Delta z)^{-2} + O(\Delta z)^2 \quad , \tag{31}$$

where $r_n = n\Delta r$, $z_j = j\Delta z$, and $\hat{L}$ is defined by

$$\hat{L}(u_j^n) = a_j^n\, b^n_{j+\frac{1}{2}}\left[a^n_{j+1}\, u^n_{j+1} - a_j^n\, u_j^n\right] + a_j^n\, b^n_{j-\frac{1}{2}}\left[a^n_{j-1}\, u^n_{j-1} - a_j^n\, u_j^n\right] \quad .$$

The Crank-Nicholson scheme for solving (30) is the one we shall use. It is simply

$$-2i\left[U_j^{n+1} - U_j^n\right] = \lambda\hat{L}\left(U_j^{n+1} + U_j^n\right) + \beta f_j^n\left(U_j^{n+1} + U_j^n\right) \quad j = 0,1,2,\ldots,N \ , \tag{33}$$

where $\lambda = (1/2)\ \Delta r/(\Delta z)^2$, $\beta = \Delta r/2$, and U_j^n is the numerical approximation of U_j^n. Equation (33) is solved in the usual fashion.

We now define the vector U^n by

$$\vec{U}^n \equiv \left(U_0^n,\ U_1^n, \ldots\ U_N^n\right)^T \quad , \tag{34}$$

where the superscript T denotes a transpose. The quantity E defined by the ℓ_2 – norm of $\vec{U}^n$, i.e.,

$$\hat{E}_n \equiv \|\vec{U}^n\|^2 \quad \sum_{\ell=0}^{N} |U_\ell^n|^2 \quad , \tag{35}$$

is the discrete analog of E defined by (27). We shall now prove that E is range independent, i.e.,

$$\hat{E}_{n+1} \equiv \hat{E}_n, \text{ for all n.} \tag{36}$$

Defining W_j by

$$W_j \equiv U_j^{n+1} + U_j^n \quad , \tag{37}$$

and multiplying (33) by W_j we obtain

$$-2i\left\{\left|U_j^{n+1}\right|^2 - \left|U_j^n\right|^2 + R_j^n = \lambda \bar{W}_j \hat{L} W_j + \beta f_j^n \left|W_j\right|^2 \quad . \tag{38}$$

The term R_j^n is real and given by

$$R_j^n \equiv -\mathrm{Im}\left[U_j^n \bar{U}_j^{n+1}\right] \quad . \tag{39}$$

Summing (38) from $j = 1$ ($z_0 = 0$) to $j = N$ ($z_N = \ell$), we obtain

$$-2i\left\{\hat{E}_{n+1} - \hat{E}_n\right\} = \lambda \sum_{j=0}^{N} \bar{W}_j \hat{L} W_j + \beta \sum_{j=0}^{N} f_j^n \left|W_j\right|^2 - \hat{R}_n \quad , \tag{40}$$

where $\hat{R}_n$ is the ℓ_2 norm of the vector $\vec{R}^n$, with components R_j^n, defined as in (34). The last two terms on the right-hand side of (40) are real. The result given in (36) follows because the term $\lambda \sum_{j=0}^{N} \bar{W}_j \hat{L} W_j$ is real also. To verify this fact we rewrite this sum as

$$\sum_{j=0}^{N} \bar{W}_j \hat{L} W_j = -g_0 + g_{N+1} - \sum_{j=0}^{N} b_{j-\frac{1}{2}} \left| a_j W_j - a_{j-1} W_{j-1} \right|^2 , \qquad (41)$$

where

$$g_0 \equiv a_0^n b_{\frac{1}{2}} \bar{W}_0 \left(a_1^n W_1 - a_0^n W_0 \right) , \qquad (42)$$

and

$$g_{n+1} \equiv a_N^n b^n_{n+\frac{1}{2}} \bar{W}_N \left(a^n_{N+\frac{1}{2}} W_{N+1} - a_N^n W_N \right) . \qquad (43)$$

Now the g_0 term is zero, because $W_0 = U_0^{n+1} + U_0^n$ and both U_0^{n+1} and U_0^n are zero. The third term in (41) is real. The term g_{N+1} vanishes. This is because the bracketed term in (43) is the discrete implementation of the boundary condition (29).

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8. CONCLUSIONS

Mathematical, physical, and computational contributions to the underwater acoustic wave propagations have been made due to combined multiple efforts among the authors presented here. These accomplishments not only enhance the PE capability by extending the solution to three-dimensional problems, but the work can be extended to handle acoustic wave propagations in elastic media. It is necessary that a complete numerical computation be performed to confirm the validity of these theoretical and computational developments.

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